Programa de Doctorado "Matemáticas"

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The Cesàro space of Dirichlet series

Author

José Jorge Bueno Contreras

Supervisors

Dr. *Guillermo P. Curbera Costello*
Dr. *Olvido Delgado Garrido*

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Contents

Introduction

This memoir is devoted to the interplay between two different mathematical objects: spaces of Dirichlet series and the sequence space ces_p.

Several spaces of Dirichlet series have been studied in recent years. Hedenmalm, Lindqvist and Seip defined in [34] the *Hilbert space of Dirichlet series*, $\mathcal{H}$, consisting of all Dirichlet series

$$f(s) := \sum_{n=1}^{\infty} a_n n^{-s}, \quad s \in \mathbb{C},$$

with square summable coefficients. They used it for solving a problem discussed by Beurling on complete sequences in the space $L^2(0,1)$. Due to the Cauchy-Schwarz inequality, each $f \in \mathcal{H}$ defines an analytic function on the vertical half-plane $\mathbb{C}_{1/2} := \{s \in \mathbb{C} : \Re(s) > 1/2\}$. The space $\mathcal{H}$ becomes a Banach space of analytic functions on $\mathbb{C}_{1/2}$ when endowed with the norm

$$\|f\|_{\mathcal{H}} := \|(a_n)_{n=1}^{\infty}\|_{\ell^2}, \quad f \in \mathcal{H}.$$

The Hardy spaces of Dirichlet series $\mathcal{H}^p$, for $1 \leq p < \infty$, were introduced by Bayart in [10]. They are analogous, in some sense, to the classical Hardy spaces $H^p(\mathbb{D})$ on the unit disc $\mathbb{D}$ of the complex plane. Namely, $\mathcal{H}^p$ is the completion of the space of Dirichlet polynomials $P(s) := \sum_{n=1}^{N} a_n n^{-s}$ for the norm

$$\|P\|_{\mathcal{H}^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p dt \right)^{1/p}.$$

The space $\mathcal{H}$ corresponds to $\mathcal{H}^p$ for $p = 2$. In [44], McCarthy considered the weighted Hilbert spaces of Dirichlet series

$$\mathcal{H}_\alpha := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=2}^{\infty} |a_n|^2 (\log n)^{-\alpha} < \infty \right\},$$

for $\alpha > 0$, endowed with the norm

$$\|f\|_{\mathcal{H}_\alpha} := \|(a_n(\log n)^{-\alpha/2})_{n=2}^{\infty}\|_{\ell^2}, \quad f \in \mathcal{H}_\alpha.$$

More recently, Bailleul and Lefèvre have studied certain classes of Bergman-type spaces of Dirichlet series, $\mathscr{A}^p_\mu$ and $\mathscr{B}^p$, for $1 \le p < \infty$, [8]. Also, another type of weighted Hilbert spaces of Dirichlet series $\mathscr{D}_\alpha$, for $\alpha > 0$, has been considered by Bailleul and Brevig in [7]. It is noticeable that the spaces $\mathcal{H}$, $\mathcal{H}^p$, $\mathcal{H}_\alpha$, $\mathscr{A}^p_\mu$, $\mathscr{B}^p$, $\mathscr{D}_\alpha$ are all Banach spaces of analytic functions on the vertical half-plane $\mathbb{C}_{1/2}$.

A particularly difficult and deep feature of Dirichlet series is their product. The pointwise product $f(s)\cdot g(s)$ of two Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ and $g(s) = \sum_{n=1}^\infty b_n n^{-s}$ is the Dirichlet series $h(s) = \sum_{n=1}^\infty c_n n^{-s}$ whose coefficients $c = (c_n)_{n=1}^\infty$ are given by the *Dirichlet convolution* $c := a \cdot b$ of the sequences $a = (a_n)_{n=1}^\infty$ and $b = (b_n)_{n=1}^\infty$, that is,

$$c_n = (a \cdot b)_n := \sum_{k|n} a_k b_{\frac{n}{k}}, \quad n \ge 1,$$

where $k|n$ denotes that k is a divisor of n.

Given a space $\mathcal{E}$ of Dirichlet series converging on a common vertical half-plane $\mathbb{C}_\theta$, a *multiplier* on $\mathcal{E}$ is an analytic function f on $\mathbb{C}_\theta$ with the property that $fg \in \mathcal{E}$ for every $g \in \mathcal{E}$. The *multiplier algebra* of $\mathcal{E}$ is the space of all multipliers on $\mathcal{E}$; we denote it by $\mathcal{M}(\mathcal{E})$. Note that $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ whenever $\mathbf{1} \in \mathcal{E}$. Neither of the spaces $\mathcal{H}$, $\mathcal{H}^p$, $\mathcal{H}_\alpha$, $\mathscr{A}^p_\mu$, $\mathscr{B}^p$, $\mathscr{D}_\alpha$ is closed under multiplication, hence a relevant question is to identify their multiplier algebras. Hedenmalm, Lindqvist and Seip identified the multiplier algebra $\mathcal{M}$ of the Hilbert space of Dirichlet series $\mathcal{H}$ proving that

$$\mathcal{M} = \mathcal{H}^\infty,$$

where $\mathcal{H}^\infty$ is the algebra of bounded analytic functions on $\mathbb{C}_0$ which can be represented as a Dirichlet series in some vertical half-plane, [34, Theorem 3.1]. This identification was a key step in solving the Beurling's question on complete sequences. It turns out that for all the spaces $\mathcal{H}^p$, $\mathcal{H}_\alpha$, $\mathscr{A}^p_\mu$, $\mathscr{B}^p$, $\mathscr{D}_\alpha$, the multiplier algebra is also the algebra $\mathcal{H}^\infty$; [10, Theorem 7], [44, Theorem 1.11] [4, Theorem 10.1 and Theorem 11.21], [7, Theorem 3].

We now turn to the sequence space ces_p. For $1 < p < \infty$, Hardy's inequality,

$$\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^\infty |a_n|^p,$$

[32, Theorem 326], established the boundedness on ℓ^p of the Cesàro averaging operator $\mathcal{C}$, defined by

$$a = (a_n)_{n=1}^\infty \longmapsto \mathcal{C}(a) := \left(\frac{1}{n} \sum_{k=1}^n a_k \right)_{n=1}^\infty .$$

The Cesàro sequence space ces_p is defined, in a natural way, as

$$ces_p := \left\{ (a_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} \ : \ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^p < \infty \right\}.$$

It is a linear space which becomes a Banach space under the norm

$$\|(a_n)_{n=1}^{\infty}\|_{ces_p} := \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^p \right)^{1/p}, \quad (a_n)_{n=1}^{\infty} \in ces_p.$$

Hardy's inequality shows that ℓ^p is continuously contained in ces_p. The contention is proper since ces_p contains sequences with arbitrarily large terms; this is an important feature of ces_p that will be detailed in Section 2.1 below. The space ces_p has been thoroughly studied by G. Bennett, [14].

In this work we study the *Cesàro space of Dirichlet series* $\mathcal{H}(ces_p)$, for $1 < p < \infty$, consisting of all Dirichlet series whose sequence of coefficients belongs to the Cesàro sequence space ces_p, that is,

$$\mathcal{H}(ces_p) := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : (a_n)_{n=1}^{\infty} \in ces_p \right\}.$$

It is a linear space, which becomes a Banach space of Dirichlet series when endowed with the norm

$$\|f\|_{\mathcal{H}(ces_p)} := \|(a_n)_{n=1}^{\infty}\|_{ces_p} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_k| \right)^p \right)^{1/p}, \quad f \in \mathcal{H}(ces_p).$$

The space $\mathcal{H}(ces_p)$ contains Dirichlet series whose coefficients can be arbitrarily large (as ces_p so does). This fact establishes a technical difference for studying $\mathcal{H}(ces_p)$, when compared with previously considered spaces of Dirichlet series. We place special emphasis on identifying the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ of $\mathcal{H}(ces_p)$.

After this Introduction and some Notation, the memoir is organized into four chapters.

Chapter 1 contains preliminary materials and is divided into four sections. The first section presents a brief account of the main works devoted to studying Banach spaces of Dirichlet series, since Hedenmalm, Lindqvist and Seip defined the Hilbert space of Dirichlet series $\mathcal{H}$ in their seminal 1997 paper, [34]. The second section collects basic results regarding Dirichlet series and their convergence properties. The third section is devoted to briefly summarize basic facts about Hardy spaces of analytic functions on the unit disc.

In the last section we present the notion of Banach lattice of sequences and its application to spaces of Dirichlet series and to Hardy spaces of analytic functions on the unit disc.

Chapter 2 deals with the study of the Cesàro space of Dirichlet series $\mathcal{H}(ces_p)$, for $1 < p < \infty$. It is organized into four sections. In the first section we review the origin, definition and properties of the sequence space ces_p. This space contains sequences with arbitrarily large terms (Example 2.1), although there is control on the growth of the partial sums (Proposition 2.2). The canonical vectors constitute a well behaved Schauder basis (Proposition 2.9). We present the isomorphic identification of the Banach dual space of ces_p given by Jagers (Theorem 2.5) and the isomorphic identification by given G. Bennett (Theorem 2.6). In the second section we define the Cesàro space of Dirichlet series $\mathcal{H}(ces_p)$ and study its basic properties. We show the existence of a well behaved Schauder basis formed by the monomials n^{-s}, for $n \geq 1$ (Proposition 2.10), and we show the approximation of functions in $\mathcal{H}(ces_p)$ by partial series obtained by restricting the number of prime numbers in the prime factorization of the summation indices (Proposition 2.11). We prove that $\mathcal{H}(ces_p)$ is a space of analytic functions on the vertical half-plane $\mathbb{C}_{1/q}$, for $1/p + 1/q = 1$, by calculating the convergence and absolute convergence abscissae (Theorem 2.12), and we study the boundedness of the point evaluation functionals on $\mathcal{H}(ces_p)$, giving sharp estimates for their norm and the precise order of growth when the abscissa approaches the critical value $1/q$ (Theorem 2.13). For the case $p = 2$ some additional information is available on the norm of the point evaluations and on equivalent expressions for the norm in $\mathcal{H}(ces_2)$. In the third section we discuss relevant Banach spaces of Dirichlet series that will appear in the memoir: $\mathcal{A}^r$ (consisting of series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $\sum_{n=1}^{\infty} |a_n| n^{-r} < \infty$), $\mathcal{A}^{2,r}$ (consisting of series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $\sum_{n=1}^{\infty} |a_n|^2 n^{-2r} < \infty$) and $\mathcal{H}^\infty(\mathbb{C}_r)$ (consisting of Dirichlet series which, possibly by analytic continuation, are bounded on the vertical half-plane $\mathbb{C}_r$). We calculate their convergence abscissae and show the boundedness of point evaluations. We prove that $\mathcal{A}^r$ is the solid core of $\mathcal{H}^\infty(\mathbb{C}_r)$ with respect to the coefficient-wise order (Proposition 2.21) and that Dirichlet series in $\mathcal{H}^\infty(\mathbb{C}_r)$ with real coefficients where the sign behaves multiplicatively in fact belong to $\mathcal{A}^r$ (Theorem 2.22). The last section exhibits other interesting spaces of Dirichlet series: $\mathcal{H}(\ell^p)$ and $\mathcal{H}([\mathcal{C}, \ell^p])$.

In Chapter 3 we look into the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ of $\mathcal{H}(ces_p)$, that is, the space of all analytic functions f (on some domain containing $\mathbb{C}_{1/q}$) such that $fg \in \mathcal{H}(ces_p)$ for all $g \in \mathcal{H}(ces_p)$. The chapter is organized in four sections. In the first section we review general results for the multiplier algebra $\mathcal{M}(\mathcal{E})$ of a space $\mathcal{E}$ of Dirichlet series. The second section is devoted to the study of the multiplier algebra of the spaces $\mathcal{A}^r$ (for which $\mathcal{M}(\mathcal{A}^r) = \mathcal{A}^r$)

and $\mathcal{A}^{2,r}$ (for which $\mathcal{M}(\mathcal{A}^{2,r}) = \mathcal{H}^\infty(\mathbb{C}_r)$). We also consider the cases of $\mathcal{M}(\mathcal{H}(\ell^p))$ and $\mathcal{M}(\mathcal{H}([\mathcal{C}, \ell^p]))$. The third section is the main section of the chapter, where we study $\mathcal{M}(\mathcal{H}(ces_p))$. We prove that monomials, and hence Dirichlet polynomials, are multipliers on $\mathcal{H}(ces_p)$ and calculate their norm as multipliers (3.4). This allows to deduce that $\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}(ces_p))$ (Theorem 3.11 and Proposition 3.3). We prove, via point evaluations, that multipliers on $\mathcal{H}(ces_p)$ are bounded functions on the vertical half-plane $\mathbb{C}_{1/q}$ (Theorem 3.11 and Proposition 3.2). This last result allows to show that the spaces $\mathcal{A}^{1/q}$, $\mathcal{M}(\mathcal{H}(ces_p))$ and $\mathcal{H}^\infty(\mathbb{C}_{1/q})$ have the same positive cone and that $\mathcal{A}^{1/q}$ is the solid core of $\mathcal{M}(\mathcal{H}(ces_p))$, with respect to the coefficient-wise order (Proposition 3.13). A fine use of the abscissae of convergence allows to deduce that $\mathcal{M}(\mathcal{H}(ces_p)) \neq \mathcal{H}^\infty(\mathbb{C}_{1/q})$ (Theorem 3.11). At this stage we know that the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ satisfies

$$\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}(ces_p)) \subsetneq \mathcal{H}^\infty(\mathbb{C}_{1/q}),$$

and that $\mathcal{A}^{1/q}$ is the solid core of $\mathcal{M}(\mathcal{H}(ces_p))$. The fact that $\mathcal{M}(\mathcal{H}(ces_p)) \neq \mathcal{H}^\infty(\mathbb{C}_{1/q})$ shows a completely different situation to that of the multipliers algebras of the previously considered spaces of Dirichlet series, $\mathcal{H}$, $\mathcal{H}^p$, $\mathcal{H}_\alpha$, $\mathscr{A}^p_\mu$, $\mathscr{B}^p$, $\mathscr{D}_\alpha$. For these spaces the multiplier algebra is $\mathcal{H}^\infty$. This fact is in accordance and actually follows from the classical result of Schur identifying the multiplier algebra of the Hardy space $H^2(\mathbb{D})$, of all Taylor series with square summable coefficients, with the space $H^\infty(\mathbb{D})$ of bounded analytic functions on the unit disc $\mathbb{D}$ of the complex plane, [47, X p.226]. However, it is noteworthy that if we consider the space of all Taylor series on the unit disc $\mathbb{D}$ with coefficients belonging to ces_p, it turns out that its multiplier algebra is not $H^\infty(\mathbb{D})$ but a rather smaller algebra, namely, the Wiener algebra of all absolutely convergent Taylor series; [24, Theorem 3.1], [25, Theorem 4.1]. This motivates the conjecture

$$\mathcal{M}(\mathcal{H}(ces_p)) = \mathcal{A}^{1/q},$$

that is, multipliers are Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ satisfying the condition $\sum_{n=1}^\infty |a_n| n^{-1/q} < \infty$. For establishing the conjecture we first prove it for multipliers $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ such that $a_n \neq 0$ only if the prime factorization of n has primes no other than the first r primes, $p_1, p_2, \ldots, p_r$, for a fixed $r \in \mathbb{N}$ (Theorem 3.15). The full identification is proven, with no restriction on the coefficients of the multipliers, in Theorem 3.17. In the last section we study the existence of compact multipliers (Theorem 3.20) and the sequence multipliers from $\mathcal{H}(ces_p)$ into its multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$, that is, we study how "close" is $\mathcal{H}(ces_p)$ from $\mathcal{M}(\mathcal{H}(ces_p))$ (Theorem 3.21).

Chapter 4 is the last one. It contains a brief discussion concerning the Cesàro averaging operator $\mathcal{C}$. The aim is to study the Cesàro averaging operator when acting on different spaces of Dirichlet series, via its action on the sequence of their coefficients. Hardy's inequality for $p = 2$ shows that $\mathcal{C}$ maps $\mathcal{H}$ into $\mathcal{H}$ boundedly and, moreover, due to the construction of the sequence space ces_2, $\mathcal{C}$ maps $\mathcal{H}(ces_2)$ into $\mathcal{H}$ boundedly. What happens, for example, for the spaces $\mathcal{H}^p$, $1 \leq p < \infty$, defined by Bayart, in the case when $p \neq 2$? For studying this last question (and other similar ones) we present an integral formula for the action of the Cesàro operator acting on Dirichlet series, (4.5) and (4.6), analogous to the classical integral formula available for the action of the Cesàro operator on Taylor series.

Results contained in this memoir are included in the following manuscripts:

- J. Bueno-Contreras, G. P. Curbera, O. Delgado, *The Cesàro space of Dirichlet series and its multiplier algebra,* submitted.
- J. Bueno-Contreras, G. P. Curbera, O. Delgado, *Multipliers on the Cesàro space of Dirichlet series,* submitted.

Notation

Throughout all the memoir we consider $1 < p < \infty$, unless explicitly stated, and denote by q its conjugate index, that is, $1/p + 1/q = 1$.

We write $\mathbb{N}$ for the set of the natural numbers $\mathbb{N} := \{1, 2, \ldots\}$. As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. Given a complex number $z \in \mathbb{C}$, its real part is written as $\Re(z)$ and its imaginary part as $\Im(z)$. For $\theta \in \mathbb{R}$, the vertical half-plane at the abscissa θ is denoted by $\mathbb{C}_\theta := \{z \in \mathbb{C} : \Re(z) > \theta\}$ and its closure by $\overline{\mathbb{C}}_\theta := \{z \in \mathbb{C} : \Re(z) \geq \theta\}$. The unit disc of the complex plane is $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and the unit circle is $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

The set of all sequences $(a_n)_{n=1}^\infty$ of real numbers is denoted by $\mathbb{R}^\mathbb{N}$, and by $\mathbb{C}^\mathbb{N}$ if the numbers are complex. The sequence of canonical vectors $(e^k)_{k=1}^\infty$ is formed by the vectors e^k whose coordinates are all zero except for the k-th coordinate which is one. The infinite-dimensional polydisc is written as $\mathbb{D}^\infty := \{z = (z_1, z_2, \ldots) : z_j \in \mathbb{D} \text{ for all } j \in \mathbb{N}\}$, and the infinite-dimensional torus as $\mathbb{T}^\infty := \{z = (z_1, z_2, \ldots) : z_j \in \mathbb{T} \text{ for all } j \in \mathbb{N}\}$.

The sign function sgn is defined by $\operatorname{sgn}(x) := 1, 0, -1$ according to $x > 0, x = 0, x < 0$. For $k, n \in \mathbb{N}$ we write $k|n$ whenever k is a divisor on n. The integer part of $x \in \mathbb{R}$, the largest integer which does no exceed x, will be denoted by $\lfloor x \rfloor$.

The Riemann zeta function ζ is given by $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ for $s \in \mathbb{C}_1$. The constant function on $\mathbb{C}$ with value one is denoted by $\mathbf{1}$.

Let $1 \leq p < \infty$. As usual, ℓ^p denotes the Banach space of sequences $(a_n)_{n=1}^\infty \in \mathbb{C}^\mathbb{N}$ such that $\|(a_n)_{n=1}^\infty\|_{\ell^p}^p := \sum_{n=1}^\infty |a_n|^p < \infty$, endowed with the norm $\|\cdot\|_{\ell^p}$. The Banach space of all bounded complex sequences is denoted by ℓ^∞ and $\|(a_n)_{n=1}^\infty\|_{\ell^\infty} := \sup_{n\geq 1} |a_n|$ for $(a_n)_{n=1}^\infty \in \ell^\infty$. The closed subspace of ℓ^∞ consisting of all sequences converging to zero is written as c_0. Given a positive measure μ on a set Ω, the space $L^p(\Omega)$ is the Banach space of all (μ-a.e. classes of) measurable complex valued functions f defined on Ω such that $\|f\|^p_{L^p(\Omega)} := \int_\Omega |f(\omega)|^p \, d\mu(\omega) < \infty$, endowed with the norm $\|\cdot\|_{L^p(\Omega)}$. The measurable functions which are bounded μ-a.e. form the Banach space $L^\infty(\Omega)$, which is endowed with the norm $\|f\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{\omega\in\Omega} |f(\omega)|$.

Chapter 1

Preliminaries

This preliminary chapter is divided into four sections. The first section presents a brief account of the main works devoted to studying Banach spaces of Dirichlet series, since Hedenmalm, Lindqvist and Seip defined the Hilbert space of Dirichlet series $\mathcal{H}$ in their seminal 1997 paper, [34]. The second section collects basic results regarding Dirichlet series and their convergence properties. The third section is devoted to briefly summarize basic facts about Hardy spaces of analytic functions on the unit disc. In the last section we present the notion of Banach lattice of sequences and its application to spaces of Dirichlet series and to Hardy spaces of analytic functions on the unit disc.

1.1 The study of Banach spaces of Dirichlet series

Although Dirichlet series (1.2) have been studied since the 19th century, function spaces consisting of Dirichlet series have not been considered until very recent times. The first spaces of this type were introduced in 1997 by Hedenmalm, Lindqvist and Seip in order to study some completeness problems in Hilbert spaces, [34]. More precisely, the initial motivation for the study of these spaces was the following problem: for every $\varphi \in L^2(0,1)$, which can be extended in a natural way to $\mathbb{R}$ as a 2-periodic odd function, denote by $\varphi_n \in L^2(0,1)$ the dilated function of φ given by

$$\varphi_n(x) := \varphi(nx), \quad n \in \mathbb{N}.$$

The only function φ whose dilated system $(\varphi_n)_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(0,1)$ is $\varphi(x) := \sqrt{2}\,\sin(\pi x)$. The question is: for which other functions

φ its dilated system forms a weaker type of basis? They focused on finding conditions under which the system $(\varphi_n)_{n=1}^{\infty}$ is a *Riesz basis*, that is, it is complete in $L^2(0,1)$ and there exist positive constants A and B such that

$$A\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{1/2} \leq \left\|\sum_{n=1}^{\infty}c_n\varphi_n\right\|_{L^2(0,1)} \leq B\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{1/2}$$

for every $(c_n)_{n=1}^{\infty} \in \ell^2$. The original question was proposed by Beurling and involves only completeness, [15]. He pointed out a possible way to study this problem consisting in the use of appropriated Dirichlet series. Considering the orthonormal basis $(\sqrt{2}\,\sin(n\pi x))_{n=1}^{\infty}$ of $L^2(0,1)$, the function φ can be written as

$$\varphi(x) = \sum_{n=1}^{\infty} a_n\sqrt{2}\sin(n\pi x), \quad \sum_{n=1}^{\infty}|a_n|^2 < \infty,$$

and then it can be associated to the Dirichlet series

$$S\varphi(s) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

Through this approach, Hedenmalm, Lindqvist and Seip found a simple characterization of Riesz basis. They also considered the problem of only completeness and gave equivalent cyclicity conditions, but they are not so fine.

Theorem 1.1 (Hedenmalm, Lindqvist, Seip)**.** *The dilated system $(\varphi_n)_{n=1}^{\infty}$ is a Riesz basis if and only if both $S\varphi$ and $1/S\varphi$ are bounded analytic functions on the vertical half-plane $\mathbb{C}_0$.*

The spaces of Dirichlet series initially introduced in [34] as part of the solution to the Riesz basis problem were $\mathcal{H}$ and $\mathcal{H}^{\infty}$. The space $\mathcal{H}$ is the space of all Dirichlet series with square-summable coefficients, that is,

$$\mathcal{H} := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=1}^{\infty}|a_n|^2 < \infty \right\}.$$

Considering the inner product of two series in $\mathcal{H}$ as the inner product of their coefficient sequences as elements of ℓ^2, we have that $\mathcal{H}$ is a Hilbert space with norm

$$\|f\|_{\mathcal{H}} := \left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}, \quad f \in \mathcal{H}.$$

From the Cauchy-Schwarz inequality it follows that

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^s}\right| \leq \left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2\Re(s)}}\right)^{1/2}$$

and so every $f \in \mathcal{H}$ converges absolutely in the vertical half-plane $\mathbb{C}_{1/2}$. Then, $\mathcal{H}$ is a space of analytic functions on $\mathbb{C}_{1/2}$, which is in fact an optimal domain (see Proposition 2.23.(a)).

The space $\mathcal{H}^\infty$ consists of all Dirichlet series convergent at some point, which define, possibly through analytic continuation, a bounded analytic function on the vertical half-plane $\mathbb{C}_0$. This space turns out to be a Banach space under the norm

$$\|f\|_{\mathcal{H}^\infty} := \sup_{s\in\mathbb{C}_0} |f(s)|, \quad f \in \mathcal{H}^\infty$$

(see [45, Theorem 6.2.1]). Although the definition of $\mathcal{H}^\infty$ does not require convergence in all $\mathbb{C}_0$ of the Dirichlet series associated to a function $f \in \mathcal{H}^\infty$, this actually happens due to the boundedness and analyticity of f and the application of a Bohr's theorem (see Theorem 1.15).

Theorem 1.2. *For every $f \in \mathcal{H}^\infty$, the associated Dirichlet series of f converges uniformly in each vertical half-plane $\mathbb{C}_\epsilon$ with $\epsilon > 0$.*

As a consequence of the previous theorem and the Carlson's identity [22] a first connection between the spaces $\mathcal{H}$ and $\mathcal{H}^\infty$ is found.

Theorem 1.3. *For every $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}^\infty$ and $\epsilon > 0$, it follows that*

$$\sum_{n=1}^\infty |a_n|^2 n^{-2\epsilon} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\epsilon+it)|^2 dt \le \|f\|^2_{\mathcal{H}^\infty}$$

In particular, $\mathcal{H}^\infty \subseteq \mathcal{H}$ continuously with $\|f\|_{\mathcal{H}} \le \|f\|_{\mathcal{H}^\infty}$ for $f \in \mathcal{H}^\infty$.

The relationship between $\mathcal{H}$ and $\mathcal{H}^\infty$ is actually deeper than the inclusion stated above, involving the concept of multiplier. A *multiplier* on $\mathcal{H}$ is an analytic function f on $\mathbb{C}_{1/2}$ such that $fg \in \mathcal{H}$ for every $g \in \mathcal{H}$. One of the main results in [34] states that the space $\mathcal{M}$ of all multipliers on $\mathcal{H}$ coincides with $\mathcal{H}^\infty$; in fact, after this result, Theorem 1.1 is proved for $S\varphi$ and $1/S\varphi$ belonging to $\mathcal{M}$.

Theorem 1.4 (Hedenmalm, Lindqvist, Seip)**.** *The equality $\mathcal{M} = \mathcal{H}^\infty$ holds with*

$$\|f\|_{\mathcal{M}} := \sup_{\substack{g\in\mathcal{H} \\ \|g\|_{\mathcal{H}}\le 1}} \|fg\|_{\mathcal{H}} = \|f\|_{\mathcal{H}^\infty}, \quad f \in \mathcal{M}.$$

Note that each $f \in \mathcal{M}$ defines a bounded multiplication operator on $\mathcal{H}$ and $\|f\|_{\mathcal{M}}$ is just its operator norm. Even more, $\mathcal{M}$ is a closed subspace of the space of all bounded linear operators on $\mathcal{H}$ (see Proposition 3.1).

Theorem 1.4 is the analogous of Schur's result which identifies the multipliers on the Hardy space $H^2(\mathbb{D})$ (see Theorem 1.24). Indeed, althoug there are other proofs (see [45, Section 6.4.3]), the original one by Hedenmalm, Lindqvist and Seip is based on the identification of the spaces $\mathcal{H}$ and $\mathcal{H}^\infty$ with suitable Hardy spaces of analytic functions. The identification is made through a technique known as the *Bohr's lift* which was introduced by Bohr in [18].

Consider a Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$. The fundamental theorem of arithmetic guarantees that each $n \in \mathbb{N}$ can be written, in an unique way, as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k},$$

where $(p_r)_{r=1}^\infty$ is the sequence of the prime numbers written in increasing order and $\alpha_r \geq 0$. Then

$$f(s) = \sum_{n=1}^\infty a_n (p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k})^{-s} = \sum_{n=1}^\infty a_n (p_1^{-s})^{\alpha_1} (p_2^{-s})^{\alpha_2} \dots (p_k^{-s})^{\alpha_k}.$$

By thinking of the prime powers p_i^{-s} as independent variables z_i, the series f can be considered as a power series in infinite variables:

$$D(f) := \sum_{n=1}^\infty a_n z_1^{\alpha_1} z_2^{\alpha_2} \dots z_k^{\alpha_k}. \tag{1.1}$$

Note that $z_i \in \mathbb{D}$ whenever $s \in \mathbb{C}_0$. One of the key points in the proof of Theorem 1.4, and probably the hardest to prove, is that the Bohr's lift establishes an isometric isomorphism between $\mathcal{H}$, $\mathcal{H}^\infty$ and the Hardy spaces on the infinite-dimensional polydisk $H^2(\mathbb{D}^\infty)$, $H^\infty(\mathbb{D}^\infty)$, respectively. After this, the problem is reduced to proving that $\mathcal{M}(H^2(\mathbb{D}^\infty)) = H^\infty(\mathbb{D}^\infty)$.

In 2002, as a generalization of $\mathcal{H}$, Bayart introduced the Hardy spaces of Dirichlet series $\mathcal{H}^p$, for $1 \leq p < \infty$, [10] and [11]. Since the norm of the Hardy space $H^p(\mathbb{D})$ is obtained by integration over circles approaching the unit circle (see Section 1.3), to obtain the analogous for Dirichlet series whose domains are vertical half-planes, it seems natural to integrate on vertical lines, just as it is done in Theorem 1.3. For P belonging to the space $\mathcal{P}$ of Dirichlet polynomials $\sum_{n=1}^N a_n n^{-s}$, set

$$\|P\|_{\mathcal{H}^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p dt \right)^{1/p}.$$

The space $\mathcal{H}^p$ is defined as the completion of $\mathcal{P}$ under the norm $\|\cdot\|_{\mathcal{H}^p}$. Using the Bohr's lift, Bayart proved that $\mathcal{H}^p$ is a Banach space isometrically isomorphic to the Hardy space on the infinite-dimensional polycircle $H^p(\mathbb{T}^\infty)$ as defined in [23] by Cole and Gamelin. After this, the following properties are obtained for $\mathcal{H}^p$: each function in $\mathcal{H}^p$ can be represented by a Dirichlet series, $\mathcal{H}^p$ is a space of analytic functions on $\mathbb{C}_{1/2}$ (which is a maximal domain) and the point evaluation δ_{s_0} at a point $s_0 \in \mathbb{C}_{1/2}$ is bounded on $\mathcal{H}^p$ with norm $\|\delta_{s_0}\| = \zeta(2\Re(s_0))^{1/p}$. Moreover, the Hardy space $\mathcal{H}^2$ is just the Hilbert space $\mathcal{H}$ and Theorem 1.4 is extended to $\mathcal{H}^p$.

Theorem 1.5 (Bayart)**.** *The space of multipliers on $\mathcal{H}^p$ coincides isometrically with $\mathcal{H}^\infty$.*

More recently, in 2014, Aleman, Olsen and Saksman proved other important result: for $1 < p < \infty$, the sequence of monomials $(n^{-s})_{n=1}^\infty$ is a Schauder basis of $\mathcal{H}^p$, [1]. This means that the partial sums of a Dirichlet series $f \in \mathcal{H}^p$ converges to f in $\mathcal{H}^p$.

In 2004, McCarthy studied the weighted Hilbert spaces

$$\mathcal{H}_{\mathbf{w}} := \left\{ f(s) = \sum_{n=n_0}^\infty a_n n^{-s} : \sum_{n=n_0}^\infty |a_n|^2 w_n < \infty \right\},$$

where $n_0 \in \mathbb{N}$ and $\mathbf{w} = (w_n)_{n=n_0}^\infty$ is a sequence of positive numbers, [44]. These spaces are Banach spaces with norm

$$\|f\|_{\mathcal{H}_{\mathbf{w}}} := \left(\sum_{n=n_0}^\infty |a_n|^2 w_n \right)^{1/2}.$$

Of course, $\mathcal{H}$ is a particular case obtained for $\mathbf{w} = (1)_{n=1}^\infty$. The sequence of weights $\mathbf{w}$ must be defined in a particular way to ensure a good behavior of $\mathcal{H}_{\mathbf{w}}$ as a space of Dirichlet series: let μ be a positive Radon measure on $[0, \infty)$ such that 0 is in the support of μ and

$$\int_0^\infty n_0^{-2\sigma} d\mu(\sigma) < \infty.$$

Then the weights in $\mathbf{w}$ are taken as

$$w_n := \int_0^\infty n^{-2\sigma} d\mu(\sigma), \quad n \geq n_0.$$

For this $\mathbf{w}$, the space $\mathcal{H}_{\mathbf{w}}$ is a space of analytic functions on $\mathbb{C}_{1/2}$, which is an optimal domain, and again the same result is obtain for its multiplier algebra, although in this time McCarthy shows a different proof without Bohr's lift.

Theorem 1.6 (McCarthy)**.** *The space of multipliers on $\mathcal{H}_{\mathbf{w}}$ coincides isometrically with $\mathcal{H}^\infty$.*

Of particular relevance is the space $\mathcal{H}_{\mathbf{w}}$ obtained for the measure μ given by $d\mu(\sigma) = 2^{\alpha}\Gamma(\alpha)^{-1}\sigma^{\alpha-1}d\sigma$ with $\alpha > 0$. In this case $\mathbf{w} = ((\log n)^{-\alpha})_{n=2}^{\infty}$ produces the space

$$\mathcal{H}_{\alpha} := \left\{ f(s) = \sum_{n=2}^{\infty} a_n n^{-s} : \sum_{n=2}^{\infty} |a_n|^2 (\log n)^{-\alpha} < \infty \right\}.$$

In 2015, Bailleul and Lefèvre introduced two Bergman-type spaces of Dirichlet series, $\mathscr{A}_{\mu}^{p}$ and $\mathscr{B}^{p}$, for $1 \leq p < \infty$, [8]. For the construcction of $\mathscr{A}_{\mu}^{p}$, they consider a probability measure μ on $(0, \infty)$ such that 0 is in the support of μ, and for P belonging to the space $\mathcal{P}$ of Dirichlet polynomials, define

$$\|P\|_{\mathscr{A}_{\mu}^{p}} := \left(\int_{0}^{\infty} \|P_{\sigma}\|_{\mathcal{H}^p}^{p} \, d\mu(\sigma) \right)^{1/p},$$

where $P_{\sigma} \in \mathcal{P}$ is given by $P_{\sigma}(s) = P(s+\sigma)$ for $s \in \mathbb{C}$. Then, they take the space $\mathscr{A}_{\mu}^{p}$ as the completion of $\mathcal{P}$ under the norm $\|\cdot\|_{\mathscr{A}_{\mu}^{p}}$. Note that $\mathscr{A}_{\mu}^{2} = \mathcal{H}_{\mathbf{w}}$ for $\mathbf{w} = (\int_{0}^{\infty} n^{-2\sigma} d\mu(\sigma))_{n=1}^{\infty}$.

For the space $\mathscr{B}^{p}$, Bailleul and Lefèvre consider the measure $A = \lambda \otimes \lambda \otimes \cdots$ on the infinite-dimensional polydisk $\mathbb{D}^{\infty}$, where λ is the normalized Lebesgue measure on $\mathbb{D}$, and for $P \in \mathcal{P}$, define

$$\|P\|_{\mathscr{B}^{p}} := \left(\int_{\mathbb{D}^{\infty}} |D(P)|^{p} dA \right)^{1/p},$$

where $D(P)$ is the analytic polynomial associated to P via (1.1). Then, $\mathscr{B}^{p}$ is the completion of $\mathcal{P}$ with respect to the norm $\|\cdot\|_{\mathscr{B}^{p}}$.

Both $\mathscr{A}_{\mu}^{p}$ and $\mathscr{B}^{p}$ turn out to be spaces of analytic functions on $\mathbb{C}_{1/2}$ which actually are Dirichlet series. Bailleul found in his thesis [4] the same result for the multiplier algebras of these spaces.

Theorem 1.7 (Bailleul)**.** *The space of multipliers on $\mathscr{A}_{\mu}^{p}$ and the space of multipliers on $\mathscr{B}^{p}$ coincide isometrically with $\mathcal{H}^{\infty}$.*

A year later, Bailleul and Brevig considered the weighted Hilbert space

$$\mathscr{D}_{\alpha} := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^{\alpha}} < \infty \right\}.$$

where $\alpha \in \mathbb{R}$ and $d(n)$ denotes the number of divisors of the integer n, [7]. Endowed with the norm

$$\|f\|_{\mathscr{D}_{\alpha}} = \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{[d(n)]^{\alpha}} \right)^{1/2},$$

$\mathscr{D}_{\alpha}$ is a Banach space of analytic functions on $\mathbb{C}_{1/2}$.

Theorem 1.8 (Bailleul, Brevig)**.** *For $\alpha > 0$, the space of multipliers on $\mathscr{D}_\alpha$ coincide isometrically with $\mathcal{H}^\infty$.*

Athough we focus our attention on the multiplier algebra, there are other important issues studied for spaces of Dirichlet series, as for instance the composition operator. In 1999, Gordon and Hedenmalm proposed and solved the following problem, [29]: for which analytic functions $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ does the composition operator $\mathcal{C}_\Phi : f \to \mathcal{C}_\Phi(f) = f \circ \Phi$ define a bounded operator from $\mathcal{H}$ into itself? They gave the following characterization for such a Φ.

Theorem 1.9 (Gordon-Hedenmalm)**.** *An analytic function $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ defines a bounded composition operator $\mathcal{C}_\Phi : \mathcal{H} \to \mathcal{H}$ if and only if*

(a) *$\Phi(s) = c_0 s + \varphi(s)$ with $c_0 \in \mathbb{N} \cup \{0\}$ and φ being Dirichlet series converging at some point, and*

(b) *Φ has an analytic extension to $\mathbb{C}_0$, also denoted by Φ, such that*

  (i) *$\Phi(\mathbb{C}_0) \subset \mathbb{C}_0$ if $c_0 \neq 0$, and*

  (ii) *$\Phi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$ if $c_0 = 0$.*

Similar results have been obtained for composition operators on $\mathcal{H}^p$ [10], $\mathscr{A}^p_\mu$ [5], $\mathscr{B}^p$, $\mathscr{D}_\alpha$ and $\mathcal{H}_\alpha$ [7]. Several properties as compactness or invertibility have also been studied in these and other papers, see [11], [12], [28], [13], [6], [4].

1.2 Dirichlet series

We include in this section a summary of basic results regarding Dirichlet series and their convergence properties. There are excellent works for further and detailed study of the subject:

- H. Bohr's doctoral dissertation, *Bidrag til de Dirichletske Rækkers Theori (Contributions to the Theory of Dirichlet Series)*, 1910, [17].
- The classical book by G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge University Press, Cambridge, 1915, [33],
- Chapter IX of the book by E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, Oxford, 1932, [49].

- Chapter 11 of the book by T. M. Apostol *Introduction to Analytic Number Theory*, Springer-Verlag, New York-Heidelberg, 1976, [2].

- The recent book by H. Queffélec and M. Queffélec *Diophantine Approximation and Dirichlet Series*, Hindustan Book Agency, New Delhi, 2013, [45].

A Dirichlet series is a series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1.2}$$

where $a = (a_n)_{n=1}^{\infty}$ is a given complex sequence, and s is a complex variable. In this work we will write $s = \sigma + it$, following Riemann's notation, where $\sigma = \Re(s)$ and $t = \Im(s)$ are real numbers.

These series are not of such importance in general analysis as power series since they only represent a very special class of analytic functions. They are, however, of great relevance in the applications of analysis to number theory. As their name implies, they were first introduced by P. G. Lejeune Dirichlet (his given name was Peter Gustav, and Lejeune Dirichlet was his surname) as part of the proof of his classical theorem on arithmetic progressions [42]. Dirichlet considered only real values for the variable s. Dirichlet series were later used by Hadamard, [30], and De la Vallée-Poussin, [26], in their proofs for the Prime Number Theorem.

According to Hardy, [33, p.1]:

> The first theorems involving complex values of s are due to Jensen, who determined the nature of the region of convergence of the general series; and the first attempt to construct a systematic theory of the function $f(s)$ was made by Cahen in a memoir which, although much of the analysis which it contains is open to serious criticism, has served–and possibly just for that reason–as the starting point of most of the later researches in the subject.

Apart from the work by Jensen [38, 39] and Cahen [21], other ideas and results regarding the convergence theory of Dirichlet series can be found in the earlier works of Dedekind [43], Kronecker [40] and Stieltjes [48].

The theory of Dirichlet series is more complicated than the theory of power series. Indeed, one of the simplest power series is $\sum_{n=0}^{\infty} z^n$ (all coefficients equal to one) which converges absolutely on the unit disc and can be analytically extended to $\mathbb{C}\backslash\{1\}$, while the corresponding Dirichlet series is $\sum_{n=1}^{\infty} n^{-s}$ which defines the Riemann zeta function!

Great differences appear in the matter of convergence regions. Power series are convergent in open discs (unless their radius of convergence is equal to 0), where they are absolutely convergent. The function defined by the series is bounded and the partial sums converge uniformly to the function on any smaller disc with the same center. The natural regions of convergence for Dirichlet series are vertical half-planes instead of discs and, unlike the case of power series, the behaviour previously described may occur in different vertical half-planes.

Theorem 1.10 (Jensen's lemma)**.** *If a Dirichlet series is convergent for $s = s_0$, then it is uniformly convergent on each cone*

$$S_C = \left\{ s : 0 \leq \frac{|s - s_0|}{\Re(s - s_0)} \leq C \right\}, \quad C \geq 1.$$

From now on, for $\theta \in \mathbb{R}$, we will denote by $\mathbb{C}_\theta$ the vertical half-plane

$$\mathbb{C}_\theta := \{s \in \mathbb{C} : \Re(s) > \theta\}.$$

An immediate consequence of the previous theorem is that any Dirichlet series convergent at some point $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ is actually convergent in all the vertical half-plane $\mathbb{C}_{\sigma_0}$, where it defines an analytic function. The following theorem directly involves convergence on vertical half-planes, although it's only pointwise convergence.

Theorem 1.11 (Bohr)**.** *Given a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, denote its partial sums by $S_N(s) = \sum_{n=1}^{N} a_n n^{-s}$. If the sequence $\{S_N(s_0)\}_{N=1}^{\infty}$ is bounded for some $s_0 \in \mathbb{C}$, then the series is convergent in the vertical half-plane $\Re(s) > \Re(s_0)$.*

Taking the previous results into account, it makes sense to define the *abscissa of convergence* of a Dirichlet series f as

$$\sigma_c(f) = \inf \{\theta : \text{the series (1.2) converges in } \mathbb{C}_\theta\}.$$

The existence of this abscissa was first noticed by Jensen in [38]. This number may be $+\infty$ if the series is nowhere convergent, or $-\infty$ if the series converges in the whole complex plane. Hence,

(a) The series (1.2) is convergent and defines an analytic function in $\mathbb{C}_{\sigma_c(f)}$.

(b) The series (1.2) is divergent in the vertical half-plane $\Re(s) < \sigma_c(f)$.

The line $\Re(s) = \sigma_c(f)$ is known as the *boundary of convergence of f*. Not much can be said about the convergence of a series on that line since it may converge or diverge in the whole line, or be convergent in some points and divergent in the rest. Note that the existence of some $s_0 \in \mathbb{C}$ such that the partial sums $\sum_{n=1}^{N} a_n n^{-s_0}$ are bounded but not convergent, can only occur for points on the boundary of convergence.

Theorem 1.12 (Cahen)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. If there exist $s_0 \in \mathbb{C}$ and $K > 0$ such that $|a_n n^{-s_0}| \leq K$ for all $n \in \mathbb{N}$, then the series converges absolutely in the vertical half-plane $\mathbb{C}_{\sigma_0+1}$, where $\sigma_0 = \Re(s_0)$.*

The previous result shows us that a somewhere convergent Dirichlet series is absolutely convergent in all vertical half-planes $\mathbb{C}_\theta$ for θ big enough. This inspires the definition of the *abscissa of absolute convergence*:

$$\sigma_a(f) = \inf \{\theta : \text{the series (1.2) converges absolutely in } \mathbb{C}_\theta\}.$$

Clearly $\sigma_c(f) \leq \sigma_a(f)$ and f converges absolutely in the vertical half-plane $\Re(s) > \sigma_a(f)$. Unlike the case of conditional convergence, if a Dirichlet series converges at any point of its *boundary of absolute convergence* then it converges on the whole line. As we mentioned at the beginning of this chapter, there may exist a region where the series is convergent but not absolutely convergent. An example of this is Riemann's alternating zeta function, also known as Dirichlet's eta function:

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}, \tag{1.3}$$

which is convergent in the vertical half-plane $\mathbb{C}_0$ but only converges absolutely in $\mathbb{C}_1$. The natural question about how big is this strip of conditional convergence is answered in the previous theorem: $\sigma_a(f) - \sigma_c(f) \leq 1$. Even more, the constant 1 cannot be improved, as it is attained in the series (1.3).

Let us denote by $\mathcal{D}$ the set of all Dirichlet series that are convergent at some point; this can be equivalently defined as the set of all Dirichlet series such that the sequence of its coefficients has, at most, a polynomial growth rate. Indeed, if $\sum_{n=1}^{\infty} a_n n^{-s_0}$ is convergent then it converges absolutely for some integer $s = N > \Re(s_0) + 1$, so $|a_n n^{-N}| \leq 1$ for n big enough, which immediately implies the polynomial growth of a_n. The sufficiency of that condition is obtained in a similar way.

Just like the case of power series, if an analytic function can be written as a Dirichlet series such representation is unique, as stated in the following theorem.

Theorem 1.13 (Dirichlet-Dedekind). *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be in $\mathcal{D}$. If there exists a sequence of complex numbers $(s_n)_{n=1}^{\infty}$ such that $\Re(s_n) \to \infty$ as $n \to \infty$, and $f(s_n) = g(s_n)$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n n^{-s}$ and $\sum_{n=1}^{\infty} b_n n^{-s}$ are identical, i.e., we have $a_n = b_n$ for all $n \in \mathbb{N}$.*

In the case of power series the Cauchy-Hadamard theorem gives us an expression for the radius of convergence of a series in terms of its coefficients. So, it is natural to look for an analogous expression for the convergence abscissa of a Dirichlet series. The corresponding formula takes slightly different forms according to whether $\sum a_n$ is convergent or not.

Theorem 1.14 (Cahen-Titchmarsh). *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. If $\sum_{n=1}^{\infty} a_n$ diverges, then*

$$\sigma_c(f) = \limsup_{N\to\infty} \frac{\log\left|\sum_{n=1}^{N} a_n\right|}{\log N}.$$

On the other hand, if $\sum_{n=1}^{\infty} a_n$ is a convergent series, we have

$$\sigma_c(f) = \limsup_{N\to\infty} \frac{\log\left|\sum_{n=N+1}^{\infty} a_n\right|}{\log N}.$$

The formula for the first case is due to Cahen, [21], and regarding the second one, which is much less known, the only reference we have found is Titchmarsh's book, [49]. Note that, if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} |a_n| n^{-s}$, then $\sigma_a(f) = \sigma_c(g)$, so

$$\sigma_a(f) = \limsup_{N\to\infty} \frac{\log \sum_{n=1}^{N} |a_n|}{\log N}$$

if $\sum_{n=1}^{\infty} |a_n| = \infty$, and

$$\sigma_a(f) = \limsup_{N\to\infty} \frac{\log \sum_{n=N+1}^{\infty} |a_n|}{\log N}$$

whenever $\sum_{n=1}^{\infty} |a_n| < \infty$.

Observe that if a Dirichlet series converges absolutely for some $s_0 \in \mathbb{C}$, the Weierstrass M-test implies its uniform convergence in the vertical half-plane $\mathbb{C}_{\sigma_0}$. Therefore, it makes sense to define the *abscissa of uniform convergence* as

$$\sigma_u(f) = \inf\left\{\theta : \text{the series (1.2) converges uniformly in } \mathbb{C}_\theta\right\},$$

and clearly $\sigma_u(f) \le \sigma_a(f)$.

Of relevance to us is also the *abscissa of regularity and boundedness* $\sigma_b(f)$, which is the infimum of those θ for which the function defined by (1.2) is analytic and bounded on $\mathbb{C}_\theta$, possibly by analytic continuation from a smaller vertical half-plane. Bohr found that the two previous behaviours actually happen in the same vertical half-plane, [19].

Theorem 1.15 (Bohr)**.** *The identity $\sigma_u(f) = \sigma_b(f)$ holds for every $f \in \mathcal{D}$.*

Bohr also found a formula for the abscissa of uniform convergence which is similar to the previous ones, although it relies on analytic properties of the function defined by the series, not only on the coefficients themselves.

Theorem 1.16 (Bohr)**.** *Given a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ such that $\sigma_u(f) \geq 0$, let $U_N := \sup_{t\in\mathbb{R}} |\sum_{n=1}^{N} a_n n^{-it}|$ for $N \geq 1$. Then*

$$\sigma_u(f) = \limsup_{N\to\infty} \frac{\log U_N}{\log N}.$$

The abscissae defined so far satisfy the inequalities

$$\sigma_c \leq \sigma_u \leq \sigma_a,$$

and the optimality of $\sigma_a - \sigma_c \leq 1$ was easily verified. The question of how big the gap between σ_u and σ_a can get required a much deeper study. It was stated by Bohr in 1913 and answered in 1931 by Bohnenblust and Hille, [16].

Theorem 1.17 (Bohnenblust-Hille)**.** *The Bohr's constant*

$$\sup_{f\in\mathcal{D}} (\sigma_a(f) - \sigma_u(f)).$$

is equal to 1/2.

A recent result shows that there is no gap between the abscissae of absolute and uniform convergence for certain kinds of Dirichlet series, [20].

Theorem 1.18 (Brevig, Heap)**.** *Suppose that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has multiplicative coefficients, that is, $a_{nm} = a_n a_m$ whenever n and m are coprime. Then $\sigma_a(f) = \sigma_b(f)$.*

Although we won't use it in this work, it is interesting to mention a fourth abscissa that does not appear so often in the literature: the holomorphy abscissa σ_h. For a Dirichlet series f, the abscissa $\sigma_h(f)$ is defined as the infimum of those $\theta \in \mathbb{R}$ such that the function defined by (1.2) admits an analytic extension to the vertical half-plane $\mathbb{C}_\theta$.

The holomorphy abscissa highlights another fundamental difference between power and Dirichlet series: whereas power series always have a singularity on its circle of convergence, a Dirichlet series may not have any singular on its boundary of convergence. An example of this is again the alternating series (1.3), which satisfies

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s} = (2^{1-s} - 1)\zeta(s), \quad s \in \mathbb{C}_0,$$

and the right side is actually an entire function, so $\sigma_h(\eta) = -\infty$. However, Dirichlet series with positive coefficients are known to always have a singularity on their convergence boundary, [41].

Theorem 1.19 (Landau)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ be such that $a_n \geq 0$ for all $n \in \mathbb{N}$. Then the analytic function defined by f has a singular point in $s = \sigma_c(f)$.*

As in the case of Fourier series, it is natural to try to recover the coefficients a_n, or its summatory function $A(x) := \sum_{n \leq x} a_n$, from the behaviour of f on some line. The corresponding result is Perron's formula.

Theorem 1.20 (Perron-Landau formula)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$. Let $\rho > \max\{0, \sigma_a(f)\}$ and $x > 1$, not an integer. Then:*

$$A(x) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} f(s) \frac{x^s}{s} ds,$$

We finish this section with the notion of convergence abscissae for sets of Dirichlet series. Those abscissae are useful as they give us the optimal vertical half-planes where the corresponding kinds of convergence occur for every element in the set. Given a set of Dirichlet series $\mathcal{E} \subseteq \mathcal{D}$, we define the abscissa of convergence of $\mathcal{E}$ as

$$\sigma_c(\mathcal{E}) = \sup_{f \in \mathcal{E}} \sigma_c(f).$$

The abscissae $\sigma_u(\mathcal{E})$ and $\sigma_a(\mathcal{E})$ are analogously defined. Balasubramanian, Calado, and H. Queffélec introduced some related notions of abscissae for spaces of Dirichlet series in [9].

1.3 Hardy spaces of analytic functions on the unit disc

The spaces $H^p(\mathbb{D})$ were introduced by F. Riesz [46], who named them in honor of Hardy after his work in [31]. The results presented in this section can be found in the classical text by Duren [27].

Given an analytic function $f : \mathbb{D} \to \mathbb{C}$ and $0 < r < 1$, the integral means M_p are defined for $1 \leq p \leq \infty$ by

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 1 \leq p < \infty,$$
$$M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|.$$

The Hardy space $H^p(\mathbb{D})$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0<r<1} M_p(r, f) < \infty.$$

It is a Banach space endowed with the norm $\|\cdot\|_{H^p(\mathbb{D})}$. Note, by the Maximum Modulus Principle, that $H^\infty(\mathbb{D})$ is the space of all bounded analytic functions on the unit disk. Also note that $H^\infty(\mathbb{D}) \subseteq H^{p_2}(\mathbb{D}) \subseteq H^{p_1}(\mathbb{D})$ whenever $1 \leq p_1 \leq p_2 \leq \infty$.

An important property of the functions in $H^p(\mathbb{D})$ is the existence of nontangential limits. Let $z \in \mathbb{T}$. For $0 < \alpha < 1$, the *nontangential approach region with vertex* z is $\Omega_\alpha(z)$ defined as the convex hull of the set $\mathbb{D}_\alpha \cup \{z\}$, where $\mathbb{D}_\alpha$ is the open disk of radius α centered at 0. A function F defined on $\mathbb{D}$ is said to *have a nontangential limit* λ *at* $z \in \mathbb{T}$ if, for each $0 < \alpha < 1$,

$$\lim_{n\to\infty} F(z_n) = \lambda$$

whenever $(z_n)_{n=1}^\infty$ is a sequence that converges to z and lies in $\Omega_\alpha(z)$.

Theorem 1.21. *Let $f \in H^p(\mathbb{D})$. Then f has a nontangential limit for almost every $z \in \mathbb{T}$. In particular*

$$f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

exists for almost every $e^{i\theta} \in \mathbb{T}$. Even more, the function f^* defined in such way belongs to $L^p(\mathbb{T})$.

The space $H^p(\mathbb{T})$ is defined then as the space of functions in $L^p(\mathbb{T})$ which are equal (a.e.) to nontangential limits of functions in $H^p(\mathbb{D})$. Considering the space $L^p(\mathbb{T})$ with the normalized Lebesgue measure on $\mathbb{T}$ and endowing $H^p(\mathbb{T})$ with the norm of $L^p(\mathbb{T})$, the spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ are isometrically isomorphic, as it is implied in the next theorem.

Theorem 1.22. *For $f \in H^p(\mathbb{D})$, the following holds*

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \int_0^{2\pi} |f^*(e^{i\theta})|^p d\theta$$

when $1 \leq p < \infty$, and, for $p = \infty$,

$$\sup_{z\in\mathbb{D}} |f(z)| = \operatorname*{ess\,sup}_{0\leq\theta<2\pi} |f^*(e^{i\theta})|.$$

The functions of $L^p(\mathbb{T})$ which are in $H^p(\mathbb{T})$ can be characterized through its Fourier coefficients.

Theorem 1.23. *Let $f \in L^p(\mathbb{T})$. Then f belongs to $H^p(\mathbb{T})$ if and only if $\hat{f}(n) = 0$ for all $n < 0$, where $\hat{f}(n)$ are the Fourier coefficients of f.*

The case $p = 2$ is special since for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$ it follows that

$$\|f\|_{H^2(\mathbb{D})} = \|(a_n)_{n=0}^{\infty}\|_{\ell^2} = \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2}.$$

Then $H^2(\mathbb{D})$ is actually a Hilbert space, with the inner product of two series being defined as the ℓ^2 product of their coefficient sequences.

The following theorem states an important connection between the spaces $H^2(\mathbb{D})$ and $H^\infty(\mathbb{D})$: the space $H^\infty(\mathbb{D})$ coincides isometrically with the multiplier algebra of $H^2(\mathbb{D})$, [47].

Theorem 1.24 (Schur)**.** *Let f be an analytic function on $\mathbb{D}$. Then $fg \in H^2(\mathbb{D})$ for every $g \in H^2(\mathbb{D})$ if and only if $f \in H^\infty(\mathbb{D})$ and*

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{\substack{g\in H^2(\mathbb{D}) \\ \|g\|_{H^2(\mathbb{D})}\leq 1}} \|fg\|_{H^2(\mathbb{D})}.$$

1.4 Banach lattices of sequences

Consider the space $\mathbb{R}^{\mathbb{N}}$ of all sequences $a = (a_n)_{n=1}^{\infty}$ with $a_n \in \mathbb{R}$ for all $n \geq 1$. The *coordinate-wise order* on $\mathbb{R}^{\mathbb{N}}$ is the partial order $\leq$ defined, for $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$, by

$$a \leq b \ \text{ precisely when } \ a_n \leq b_n \text{ for all } n \geq 1.$$

Given $a, b \in \mathbb{R}^{\mathbb{N}}$, the supremum and the infimum of a and b are, respectively,

$$a \vee b = (\max\{a_n, b_n\})_{n=1}^{\infty}, \quad a \wedge b = (\min\{a_n, b_n\})_{n=1}^{\infty}.$$

The modulus of $a \in \mathbb{R}^{\mathbb{N}}$ is $|a| = a \vee (-a) = (|a_n|)_{n=1}^{\infty}$.

A *real Banach sequence space* is a linear subspace $E \subseteq \mathbb{R}^{\mathbb{N}}$ endowed with a complete norm $\|\cdot\|_E$. We say that E is a *real Banach lattice* (of sequences) if, in addition, it satisfies that

(1) $a \vee b, a \wedge b \in E$ for every $a, b \in E$, and

(2) $\|a\|_E \leq \|b\|_E$ whenever $a, b \in E$ with $|a| \leq |b|$ (monotony of the norm).

It follows that $\|a\|_E = \|\,|a|\,\|_E$ for all $a \in E$.

For a subset $A \subseteq \mathbb{R}^{\mathbb{N}}$, the *positive cone* of A is the set A^+ of all $a \in A$ such that $a \geq 0$, that is, $a_n \geq 0$ for all $n \geq 1$. The set A is *solid* (for the coordinate-wise order) if the condition $a, b \in \mathbb{R}^{\mathbb{N}}$ with $a \in A$ and $|b| \leq |a|$ implies that $b \in A$. The *solid core* of A is the largest solid linear subspace contained in A.

Consider the space $\mathbb{C}^{\mathbb{N}}$ of all sequences $a = (a_n)_{n=1}^{\infty}$ with $a_n \in \mathbb{C}$ for all $n \geq 1$. A *complex Banach lattice* (of sequences) is

$$E_{\mathbb{C}} := E + iE \subseteq \mathbb{C}^{\mathbb{N}},$$

where $E \subseteq \mathbb{R}^{\mathbb{N}}$ is a real Banach lattice (of sequences). The modulus of $a \in E_{\mathbb{C}}$ is taken as $|a| := (|a_n|)_{n=1}^{\infty}$, which turns out to be an element of E^+. Then, the norm in $E_{\mathbb{C}}$ is defined by $\|a\|_{E_{\mathbb{C}}} := \|\,|a|\,\|_E$. For every $a, b \in E_{\mathbb{C}}$ with $|a| \leq |b|$ it follows that $\|a\|_{E_{\mathbb{C}}} \leq \|b\|_{E_{\mathbb{C}}}$.

The positive cone of $A \subseteq \mathbb{C}^{\mathbb{N}}$ is $A^+ := A \cap (\mathbb{R}^{\mathbb{N}})^+$. We say that A is *solid* if $b \in A$ whenever $b \in \mathbb{C}^{\mathbb{N}}$ with $|b| \leq |a|$ and $a \in A$. The *solid core* of A is the largest solid linear subspace contained in A.

Let $a = (a_n)_{n=1}^{\infty}$ and $a^m = (a_n^m)_{n=1}^{\infty}$, for $m \geq 1$, be real sequences. We write $a^m \uparrow a$ if, for each fixed n-th coordinate, the sequence $(a_n^m)_{m=1}^{\infty}$ is increasing and $\sup_{m \geq 1} a_n^m = a_n$. Similarly, $a^m \downarrow a$ if, for each fixed n-th coordinate, the sequence $(a_n^m)_{m=1}^{\infty}$ is decreasing and $\inf_{m \geq 1} a_n^m = a_n$. A complex Banach lattice of sequences E is said to be *order continuous* if for every $(a^m)_{m=1}^{\infty} \subset E^+$ such that $a^m \downarrow 0$ it follows that $\lim_{m \to \infty} \|a^m\|_E = 0$.

These notions (lattice, positive cone, solid set, solid core, order continuity) can be transferred to a space of analytic functions M via the *coefficient-wise* *order*, which is the coordinate-wise order in the sequence space of coefficients of the functions belonging to M. The following two examples are of interest to us.

Let $H(\mathbb{D})$ be the space of all analytic functions on the unit disc $\mathbb{D}$ of the complex plane. A Banach space of analytic functions on $\mathbb{D}$ is a linear subspace $X \subseteq H(\mathbb{D})$ endowed with a complete norm $\|\cdot\|_X$. The coefficient-wise order in X corresponds to the coordinate-wise order in the associated sequence space

$$\widehat{X} := \left\{ (a_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} a_n z^n \in X \right\}.$$

In a similar way, a Banach space of converging Dirichlet series $\mathcal{E}$ is a linear subspace $\mathcal{E} \subseteq \mathcal{D}$ endowed with a complete norm $\|\cdot\|_{\mathcal{E}}$. The coefficient-wise order in $\mathcal{E}$ corresponds to the coordinate-wise order in the associated sequence space

$$\widehat{\mathcal{E}} := \left\{ (a_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{E} \right\},$$

For matters related to complex Banach lattices, see Section 91 in Chapter 12 of the book *Riesz Spaces II* by A. C. Zaanen, [51].

Chapter 2

The Cesàro space of Dirichlet series

This chapter is devoted to the study of the Cesàro space of Dirichlet series $\mathcal{H}(ces_p)$, for $1 < p < \infty$. It is organized into three sections. In the first section we review the origin, definition and properties of the sequences space ces_p. This space contains sequences with arbitrarily large terms (Example 2.1), although there is control on the growth of the partial sums (Proposition 2.2). The canonical vectors constitute a well behaved Schauder basis (Proposition 2.9). We present the isomorphic identification of the Banach dual space given by Jagers (Theorem 2.5) and the isomorphic identification by given G. Bennett (Theorem 2.6). In the second section we define the Cesàro space of Dirichlet series $\mathcal{H}(ces_p)$ and study its basic properties. We show the existence of a well behaved Schauder basis formed by the monomials n^{-s}, for $n \geq 1$ (Proposition 2.10), and we show the approximation of functions in $\mathcal{H}(ces_p)$ by partial series obtained by restricting the number of prime numbers in the summation indices (Proposition 2.11). We show that $\mathcal{H}(ces_p)$ is a space of analytic functions on the vertical half-plane $\mathbb{C}_{1/q}$, for $1/p + 1/q = 1$, by calculating the convergence and absolute convergence abscissae (Theorem 2.12), and we study the boundedness of the point evaluation functionals on $\mathcal{H}(ces_p)$, giving sharp estimates for their norm and the precise order of growth when the abscissa approaches the critical value $1/q$ (Theorem 2.13). For the case $p = 2$ some additional information is available on the norm of the point evaluations and on equivalent expressions for the norm in $\mathcal{H}(ces_2)$. In the third section we discuss other Banach spaces of Dirichlet series that will appear in the memoir: $\mathcal{A}^r$, $\mathcal{A}^{2,r}$ and $\mathcal{H}(\mathbb{C}_r)$. We calculate their convergence abscissae and prove the boundedness of point evaluations. We prove that $\mathcal{A}^r$ is the solid core of $\mathcal{H}(\mathbb{C}_r)$ (Proposition 2.21) and that Dirichlet series in $\mathcal{H}(\mathbb{C}_r)$ with real coefficients where the sign behaves multiplicatively in fact belong to $\mathcal{A}^r$

(Theorem 2.22). We end exhibiting the spaces of Dirichlet series $\mathcal{H}(\ell^p)$ and $\mathcal{H}([\mathcal{C},\ell^p])$.

2.1 The Cesàro sequence space ces_p

The Cesàro sequence space ces_p was first introduced in 1968 when the Dutch Mathematical Society proposed the identification of its dual space as a prize problem, [50]. Jagers solved the problem in a more general context, and gave an explicit isometric identification of the dual space, [37]. In 1996, G. Bennett made a thorough study of the spaces ces_p in the memoire entitled *"Factorizing the Classical Inequalities"*, [14]. Astashkin and Maligranda have also studied these spaces, see, for example, [3], and the references therein.

For $1 < p < \infty$, the Cesàro sequence space ces_p is defined as

$$ces_p := \left\{ x = (x_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} \ : \ \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}.$$

It is a linear space which becomes a Banach space under the norm

$$\|x\|_{ces_p} := \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}, \qquad x \in ces_p.$$

Note that $1 < p_1 \leq p_2 < \infty$ implies $ces_{p_1} \subseteq ces_{p_2}$.

The space ces_p arises in a natural way from *Hardy's inequality*,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} |x_n|^p,$$

[32, Theorem 326], which establishes the boundedness on ℓ^p of the Cesàro averaging operator $\mathcal{C}$ defined by

$$x = (x_n)_{n=1}^{\infty} \longmapsto \mathcal{C}(x) := \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n=1}^{\infty}.$$

Hardy's inequality shows that ℓ^p is continuously contained in ces_p. The averaging process appearing in the definition of ces_p "softens" the growth of sequences, allowing some unbounded sequences to belong to ces_p. In fact, ces_p contains sequences with arbitrarily large terms.

Example 2.1. Given any complex sequence $a = (a_m)_{m=1}^{\infty}$ and an increasing sequence of natural numbers $1 < N_1 < N_2 < \ldots$, define

$$x_k := \begin{cases} a_m & \text{if} \quad k = N_m \\ 0 & \text{otherwise} \end{cases}.$$

It is possible to choose a sequence $(N_k)_{k=1}^{\infty}$ so that $x = (x_k)_{k=1}^{\infty}$ belongs to ces_p. Indeed, from the definition of x it follows that

$$\begin{aligned} \sum_{k=N_m}^{N_{m+1}-1} \frac{1}{k^p} \left(\sum_{j=1}^{k} |x_j| \right)^p &= \sum_{k=N_m}^{N_{m+1}-1} \frac{1}{k^p} \left(\sum_{i=1}^{m} |a_i| \right)^p \\ &\leq \left(\sum_{i=1}^{m} |a_i| \right)^p \sum_{k=N_m}^{\infty} \frac{1}{k^p} \\ &\leq \left(\sum_{i=1}^{m} |a_i| \right)^p \frac{1}{p-1} \cdot \frac{1}{(N_m - 1)^{p-1}}. \end{aligned}$$

Taking $(N_k)_{k=1}^{\infty}$ increasing fast enough, in a way that the last term of the above inequality is appropriately small, we have that

$$\sum_{m=1}^{\infty} \frac{1}{k^p} \left(\sum_{j=1}^{k} |x_j| \right)^p = \sum_{m=1}^{\infty} \sum_{k=N_m}^{N_{m+1}-1} \frac{1}{k^p} \left(\sum_{j=1}^{k} |x_j| \right)^p < \infty.$$

The previous example is specially interesting when the sequence a is not bounded, showing us that the space ces_p is much larger than ℓ^p. Nevertheless there is still some control on the growth of the partial sums of elements in ces_p, as the next result shows, [25, Proposition 2.3(iii)]. We include a proof for completeness. Recall that we denote by q the conjugate exponent of p.

Proposition 2.2 (Curbera, Ricker)**.** *If $(x_n)_{n=1}^{\infty} \in ces_p$ then*

$$\lim_{n \to \infty} \frac{1}{n^{1/q}} \sum_{k=1}^{n} |x_k| = 0.$$

Proof. We have that

$$\begin{aligned} \sum_{k=n}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{k} |x_j| \right)^p &\geq \sum_{k=n}^{\infty} \left(\frac{1}{k} \sum_{j=1}^{n} |x_j| \right)^p \\ &= \left(\sum_{j=1}^{n} |x_j| \right)^p \sum_{k \geq n} \frac{1}{k^p} \\ &\geq \left(\sum_{j=1}^{n} |x_j| \right)^p \frac{1}{(p-1)n^{p-1}}. \end{aligned}$$

Since the first term of the above inequality tends to zero when $n \to \infty$ as $(x_n)_{n=1}^{\infty} \in ces_p$, it follows that

$$\lim_{n\to\infty} \frac{1}{n^{p-1}} \left(\sum_{j=1}^{n} |x_j| \right)^p = 0.$$

Raising to the power $1/p$, as $(p-1)/p = 1/q$, we obtain the result. $\square$

Jagers studied in [37] a family of spaces more general than ces_p, namely the spaces b_p, for $1 \leq p < \infty$. Given a sequence $\beta = (\beta_n)_{n=1}^{\infty}$ of positive numbers, the space b_p associated to β consists of all sequences $x = (x_n)_{n=1}^{\infty}$ satisfying

$$\|x\|_{b_p} := \left(\sum_{n=1}^{\infty} \left(\beta_n \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty.$$

The particular case of $\beta = (n^{-1})_{n=1}^{\infty}$ gives the Cesàro sequence space ces_p. It is immediate that $b_p = \{0\}$ if and only if $\sum_{n=1}^{\infty} \beta_n^p = \infty$. Hence, it is natural to assume that $\sum_{n=1}^{\infty} \beta_n^p < \infty$. We denote

$$B_m := \sum_{k=m}^{\infty} \beta_k^p, \quad m \geq 1.$$

The space b_p is a Banach space endowed with the norm $\|\cdot\|_{b_p}$.

Proposition 2.3. *For each $m \geq 1$, the m-th coordinate functional π_m on b_p, given by $\pi_m(x) := x_m$ for $x = (x_n)_{n=1}^{\infty} \in b_p$, is bounded and*

$$\|\pi_m\| = B_m^{-\frac{1}{p}}.$$

Proof. For every $x = (x_n)_{n=1}^{\infty} \in b_p$ we have that

$$\|x\|_{b_p}^p \geq \sum_{n=m}^{\infty} \left(\beta_n \sum_{k=1}^{n} |x_k| \right)^p \geq |x_m|^p B_m$$

and so $|\pi_m(x)| \leq \|x\|_{b_p} B_m^{-\frac{1}{p}}$, which gives $\|\pi_m\| \leq B_m^{-\frac{1}{p}}$. On the other hand, note that the canonical vectors $(e^n)_{n=1}^{\infty}$ are included in b_p with $\|e^n\|_{b_p} = B_n^{\frac{1}{p}}$. Then, taking $x = e^m/\|e^m\|_{b_p}$ we obtain that $B_m^{-\frac{1}{p}} = |\pi_m(x)| \leq \|\pi_m\|$. $\square$

General properties of b_p are collected in the next proposition, see [37].

Proposition 2.4 (Jagers)**.** *The following statements hold:*

(a) *For every* $x = (x_n)_{n=1}^\infty \in b_p$ *it follows that* $\sum_{n=1}^N x_n e^n$ *converges to* x *in the norm of* b_p. *In particular, as the m-th coordinate functionals are continuous, the canonical vectors* $(e^n)_{n=1}^\infty$ *form a Schauder basis.*

(b) *The dual Banach space* b_p^* *of* b_p *can be identified with the sequence space*

$$\left\{(u_n)_{n=1}^\infty \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^\infty |x_n u_n| < \infty \text{ for all } (x_n)_{n=1}^\infty \in b_p\right\}, \tag{2.1}$$

via $u^* \in b_p^* \mapsto (u^*(e^n))_{n=1}^\infty$.

(c) *For* $1 < p < \infty$, *the space* b_p *is reflexive.*

The space described in (2.1) is known as the *Köthe dual space* of b_p. In general, the Köthe dual of a Banach sequence space does not coincide with its dual Banach space. The main result of [37] is the following isometric characterization of the dual space b_p^* of b_p.

Theorem 2.5 (Jagers). *For* $u = (u_n)_{n=1}^\infty \in c_0$, *adopting the convention* $u_\infty = B_\infty = 0$, *set*

(a) $m(1) := \max\{k \in \mathbb{N} \cup \{\infty\} : |u_k| = \max_{j\geq 1} |u_j|\}$.

(b) *For* $n \geq 1$, *provided* $m(n)$ *is defined and finite,*

$$m(n+1) := \max\left\{k \in \mathbb{N}\cup\{\infty\} \;:\; k > m(n),\right.$$
$$\left.\frac{|u_{m(n)}| - |u_k|}{B_{m(n)} - B_k} = \min_{m(n)<j\leq\infty} \frac{|u_{m(n)}| - |u_j|}{B_{m(n)} - B_j}\right\},$$

in other case $m(n+1)$ *is not defined.*

(c) $D(u)$ *is the set of all* $k \geq 1$ *such that* $m(k)$ *is defined and finite.*

(d) *For* $p > 1$,

$$_q\|u\| := \left(\sum_{n\in D(u)} \left(\frac{|u_{m(n)}| - |u_{m(n+1)}|}{B_{m(n)} - B_{m(n+1)}}\right)^q \left(B_{m(n)} - B_{m(n+1)}\right)\right)^{1/q},$$

and for $p = 1$,

$$_\infty\|u\| := \sup_{n\in D(u)} \frac{|u_{m(n)}| - |u_{m(n+1)}|}{B_{m(n)} - B_{m(n+1)}}.$$

Define ${}_qd := \{u \in c_0 : \ {}_q\|u\| < \infty\}$. *Then* b_p^* *is isometrically isomorphic to* ${}_qd$ *which turns out to be a Banach space with norm* ${}_q\| \cdot \|$.

It must be noted that although Jagers considered real sequences, all the results remain valid for complex sequences, as $(b_p)_{\mathbb{R}} := b_p \cap \mathbb{R}^{\mathbb{N}}$ is a solid real Banach lattice of sequences and so b_p is the solid complex Banach lattice $b_p = (b_p)_{\mathbb{R}} + i(b_p)_{\mathbb{R}}$ (see Section 1.4 in the Preliminaries).

Observe that the exact calculation of ${}_q\|u\|$ can be quite involved. For the particular case of ces_p, Bennett gave a simpler, though just isomorphic, identification of the dual space ces_p^*, [14, p.61]. For $1 < p < \infty$, consider the Banach space $d(p)$ consisting of all complex sequences $u = (u_n)_{n=1}^\infty$ satisfying

$$\|u\|_{d(p)} := \left(\sum_{n=1}^{\infty} \sup_{k \geq n} |u_k|^p \right)^{1/p} < \infty. \tag{2.2}$$

Given a sequence $(u_n)_{n=1}^\infty$, the sequence $(\tilde{u}_n)_{n=1}^\infty$ defined by

$$\tilde{u}_n := \sup_{k \geq n} |u_k|, \quad n \geq 1,$$

is known as the *least decreasing majorant* of $(u_n)_{n=1}^\infty$.

Theorem 2.6 (Bennett)**.** *For* $1 < p < \infty$ *and* $1/p + 1/q = 1$, *the dual space* ces_p^* *of* ces_p *can be identified with the sequence space* $d(q)$. *The relationship between their norms is given by*

$$\frac{1}{q}\|u\|_{d(q)} \leq \|u\|_{ces_p^*} \leq (p-1)^{1/p}\|u\|_{d(q)}. \tag{2.3}$$

Regarding equivalent expressions for the norm in ces_p, the following is of interest. The Copson spaces $cop(p)$, for $0 < p < \infty$, defined by

$$cop(p) := \left\{ x = (x_k)_{k=1}^\infty \in \mathbb{C}^{\mathbb{N}} : \ \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k} \right)^p < \infty \right\},$$

were studied by Bennett in [14]. When $p \geq 1$, they are Banach sequence spaces under the norm

$$\|x\|_{cop(p)} := \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k} \right)^p \right)^{1/p}, \quad x \in cop(p).$$

Bennett proved, for $1 < p < \infty$, that $cop(p)$ and ces_p are actually the same set, and gave an equivalence for their norms, [14, §10 and Theorem 10.8].

Theorem 2.7 (Bennett). *For $1 < p < \infty$ we have that $ces_p = cop(p)$ and*

$$\frac{1}{p(p-1)^{1/p}} \|x\|_{cop(p)} \leq \|x\|_{ces_p} \leq \frac{p}{p-1} \|x\|_{cop(p)} .$$

Optimal values for the constants are, for $1 < p \leq 2$,

$$\|x\|_{cop(p)} \leq (p-1)^{1/p} \|x\|_{ces_p} ,$$

and, for $2 \leq p < \infty$,

$$\|x\|_{ces_p} \leq \zeta(p)^{1/p} \|x\|_{cop(p)} .$$

The next equivalence, due to Bennett, [14, Theorem 11.5], is inspired by a celebrated inequality of Hilbert, [32, Chapter IX], which asserts that

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{|x_k|}{n+k-1} \right)^p \leq \left(\frac{\pi}{\sin(\pi/p)} \right)^p \sum_{k=1}^{\infty} |x_k|^p$$

where $1 < p < \infty$, and the inequality is strict unless $x_n = 0$ for all n. The space $hil(p)$ is then defined to be the set of all sequences $x = (x_n)_{n=1}^{\infty}$ for which

$$\|x\|_{hil(p)} = \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{|x_k|}{n+k-1} \right)^p \right)^{1/p} < \infty.$$

Theorem 2.8 (Bennett). *For $1 < p < \infty$ we have that $hil(p) = ces_p$ and*

$$\|x\|_{ces_p} \leq \|x\|_{hil(p)} \leq \frac{\pi}{q \sin(\pi/p)} \|x\|_{ces_p},$$

where the constants are the best possible.

Other properties of the standard basis $(e_n)_{n=1}^{\infty}$ were observed in [25, Proposition 2.1].

Proposition 2.9. *The sequence of canonical vectors $(e^n)_{n=1}^{\infty}$ is an unconditional, boundedly complete and shrinking Schauder basis for ces_p.*

As noted for the spaces b_p, the Cesàro space ces_p is a solid complex Banach lattice for the coordinate-wise order. In particular, it satisfies that given complex sequences $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$ such that $|b_n| \leq |a_n|$ for all $n \geq 1$, if $a \in ces_p$ then $b \in ces_p$ and $\|b\|_{ces_p} \leq \|a\|_{ces_p}$. Other important lattice property of ces_p is the order continuity: for every $(a^m)_{m=1}^{\infty} \subset ces_p^+$ such that $a^m \downarrow 0$ it follows that $\|a^m\|_{ces_p} \to 0$ as $m \to \infty$. This fact follows (in the same way as for ℓ^p) from applying the Dominated Convergence Theorem to the functions f_m defined on $\mathbb{N}$ by $f_m(n) = (\frac{1}{n}\sum_{k=1}^n a_k^m)^p$, where $a^m = (a_n^m)_{n=1}^\infty$. Denoting by λ the counting measure on $\mathbb{N}$, we have that $f_m \in L^1(\lambda)$ and $\|f_m\|_{L^1(\lambda)} = \|a^m\|_{ces_p}^p$ for all $m \geq 1$. Also, we can apply directly the fact that ℓ^p is order continuous. Since ces_p is solid, its order continuity can be rewritten as: for every $(a^m)_{m=1}^\infty \subset ces_p \cap \mathbb{R}^\mathbb{N}$ such that $a^m \uparrow a$ and $|a^m| \leq b$ for all $m \geq 1$ with $b \in ces_p^+$ it follows that $a \in ces_p$ and $a^m \to a$ in ces_p as $m \to \infty$.

2.2 The Cesàro space of Dirichlet series $\mathcal{H}(ces_p)$

For $1 < p < \infty$, the Cesàro space of Dirichlet series $\mathcal{H}(ces_p)$ consists of all Dirichlet series whose sequence of coefficients belongs to the Cesàro space ces_p, that is,

$$\mathcal{H}(ces_p) := \left\{ f(s) = \sum_{n=1}^\infty a_n n^{-s} : \sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^p < \infty \right\}.$$

It is a linear space which becomes a Banach space of Dirichlet series when endowed with the norm

$$\|f\|_{\mathcal{H}(ces_p)} := \left(\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^p \right)^{1/p}, \quad f \in \mathcal{H}(ces_p). \tag{2.4}$$

Note that $1 < p_1 \leq p_2 < \infty$ implies $\mathcal{H}(ces_{p_1}) \subseteq \mathcal{H}(ces_{p_2})$.

The space $\mathcal{H}(ces_p)$ inherits many of its functional properties from the sequence space ces_p, as they are isometrically isomorphic. Since the canonical vectors $\{e^m : m \geq 1\}$ of ces_p correspond to the monomials $\{m^{-s} : m \geq 1\}$ in $\mathcal{H}(ces_p)$, from Propositions 2.4 and 2.9 we have the following result.

Proposition 2.10. *The following statements hold:*

(a) For every $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}(ces_p)$ it follows that the Dirichlet polynomials $\sum_{n=1}^N a_n n^{-s}$ converges to f in the norm of $\mathcal{H}(ces_p)$. Moreover, from the monotony of the norm of ces_p,

$$\|f\|_{\mathcal{H}(ces_p)} = \sup_{N \in \mathbb{N}} \left\| \sum_{n=1}^N a_n n^{-s} \right\|_{\mathcal{H}(ces_p)}.$$

(b) The sequence of monomials $\{m^{-s} : m \geq 1\}$ is an unconditional, boundedly complete and shrinking Schauder basis for $\mathcal{H}(ces_p)$. In particular, $\mathcal{H}(ces_p)$ is reflexive.

A further approximation for functions in $\mathcal{H}(ces_p)$ is possible. Let $(p_k)_{k=1}^{\infty}$ denote the sequence of the prime numbers written in increasing order. For $r \in \mathbb{N}$, let

$$\mathbb{N}_r := \Big\{ n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r} : \ \alpha_1, \dots, \alpha_r \geq 0 \Big\}.$$

Consider the map Q_r defined by

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \mapsto Q_r(f) := \sum_{n \in \mathbb{N}_r} a_n n^{-s}.$$

The map Q_r is in fact a projection

$$Q_r \colon \mathcal{H}(ces_p) \to \mathcal{H}(ces_p).$$

An remarkable property of the projection Q_r is its multiplicativity, namely,

$$Q_r(fg) = Q_r(f) Q_r(g),$$

which holds for any pair of Dirichlet series f and g, see [45, p.157].

The order continuity of ces_p and the monotony of its norm give the following result.

Proposition 2.11. *For each $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathcal{H}(ces_p)$ it follows that $\sum_{n \in \mathbb{N}_r} a_n n^{-s}$ converges to f in the norm of $\mathcal{H}(ces_p)$. Moreover,*

$$\|f\|_{\mathcal{H}(ces_p)} = \sup_{r \in \mathbb{N}} \left\| \sum_{n \in \mathbb{N}_r} a_n n^{-s} \right\|_{\mathcal{H}(ces_p)}.$$

Since ces_p is a solid sequence space for the coordinate-wise order, $\mathcal{H}(ces_p)$ is a solid space of Dirichlet series for the coefficient-wise order.

We next show that $\mathcal{H}(ces_p)$ is a Banach space of analytic functions. For this, we determine the abscissa of convergence and the abscissa of absolute convergence of $\mathcal{H}(ces_p)$.

Theorem 2.12. *Every Dirichlet series $f \in \mathcal{H}(ces_p)$ converges, in fact absolutely, on the vertical half-plane $\mathbb{C}_{1/q}$, where q is the conjugate exponent of p. Moreover, the value $1/q$ cannot be improved, that is,*

$$\sigma_c(\mathcal{H}(ces_p)) = \sigma_a(\mathcal{H}(ces_p)) = 1/q.$$

Consequently, $\mathcal{H}(ces_p)$ is a Banach space of analytic functions on $\mathbb{C}_{1/q}$, which is a maximal domain.

Proof. Let $f \in \mathcal{H}(ces_p)$ with $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $(a_n)_{n=1}^{\infty} \in ces_p$. Set $r > 1/q$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|a_n|}{n^r} &\leq r \sum_{n=1}^{\infty} |a_n| \sum_{k=n}^{\infty} \frac{1}{k^{r+1}} \\
&= r \sum_{k=1}^{\infty} \frac{1}{k^{r+1}} \sum_{n=1}^{k} |a_n| \\
&\leq r \left(\sum_{k=1}^{\infty} \frac{1}{k^{rq}} \right)^{1/q} \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{n=1}^{k} |a_n| \right)^p \right)^{1/p} \\
&= r\zeta(rq)^{1/q} \, \|f\|_{\mathcal{H}(ces_p)}.
\end{aligned}$$

Then $\sigma_a(f) \leq 1/q$ for all $f \in \mathcal{H}(ces_p)$ and so $\sigma_a(\mathcal{H}(ces_p)) \leq 1/q$.

On the other hand, for $r > 1/p$ set $f(s) := \sum_{n=1}^{\infty} n^{-(r+s)}$. Note that $f \in \mathcal{H}(ces_p)$ as $(n^{-r})_{n=1}^{\infty} \in \ell^p \subseteq ces_p$. Since $f(s) = \zeta(r+s)$, it follows that $\sigma_a(f) = 1 - r$ which tends to $1/q$ as $r \to 1/p$. Thus, $\sigma_c(\mathcal{H}(ces_p)) \geq 1/q$.

The conclusion follows since $\sigma_c(\mathcal{H}(ces_p)) \leq \sigma_a(\mathcal{H}(ces_p))$. $\square$

We study next the boundedness of the linear functional δ_{s_0} on $\mathcal{H}(ces_p)$ given by evaluation at a point $s_0 \in \mathbb{C}_{1/q}$:

$$f \in \mathcal{H}(ces_p) \mapsto \delta_{s_0}(f) := f(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} \in \mathbb{C}.$$

Note, for $s_0 = \sigma + it \in \mathbb{C}_{1/q}$ and $f \in \mathcal{H}(ces_p)$ with $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, that the proof of Theorem 2.12 already shows that

$$|\delta_{s_0}(f)| \leq \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \leq \sigma\zeta(\sigma q)^{1/q} \|f\|_{\mathcal{H}(ces_p)}.$$

Thus, δ_{s_0} belongs to the dual space $\mathcal{H}(ces_p)^*$ of $\mathcal{H}(ces_p)$ with

$$\|\delta_{s_0}\| \leq \sigma\zeta(\sigma q)^{1/q}. \tag{2.5}$$

We provide sharp estimates for the norm $\|\delta_{s_0}\|$, the precise order of growth when the abscissa approaches from the right the critical value $1/q$, and the asymptotic value when the abscissa increases to ∞. For this we require the identifications of the dual Banach space of ces_p given by Jagers and Bennett, see Theorems 2.5 and 2.6.

Theorem 2.13. *For each $s_0 = \sigma + it \in \mathbb{C}_{1/q}$ the linear functional δ_{s_0} is bounded on $\mathcal{H}(ces_p)$, and*

$$\frac{1}{q}\zeta(\sigma q)^{1/q} \leq \|\delta_{s_0}\| \leq (p-1)^{1/p}\zeta(\sigma q)^{1/q}.$$

Moreover, there is a value σ_p, depending only on p, such that $\|\delta_{s_0}\| = \zeta(p)^{-1/p}$ whenever $\sigma \geq \sigma_p$.

Proof. Let $s_0 \in \mathbb{C}_{1/q}$. For $f(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0}$ in $\mathcal{H}(ces_p)$ with $(a_n)_{n=1}^{\infty} \in ces_p$, we write

$$\delta_{s_0}(f) = f(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} = \left\langle \left(n^{-s_0}\right)_{n=1}^{\infty}, (a_n)_{n=1}^{\infty} \right\rangle_{ces_p},$$

where the duality is between $(a_n)_{n=1}^{\infty} \in ces_p$ and $(n^{-s_0})_{n=1}^{\infty} \in ces_p^*$. Thus, δ_{s_0} acting on $\mathcal{H}(ces_p)$ can be identified with the sequence $(n^{-s_0})_{n=1}^{\infty}$ acting on ces_p. Since $\mathcal{H}(ces_p)$ and ces_p are isometric, we have that the norms of δ_{s_0} and $(n^{-s_0})_{n=1}^{\infty}$ as elements of ces_p^* are equal. Using Bennett's identification of ces_p^* as the space $d(q)$, from (2.3), it follows that

$$\frac{1}{q}\|(n^{-s_0})_{n=1}^{\infty}\|_{d(q)} \leq \|\delta_{s_0}\| \leq (p-1)^{1/p}\|(n^{-s_0})_{n=1}^{\infty}\|_{d(q)}.$$

Note, from (2.2), that for sequences $(d_n)_{n=1}^{\infty}$ such that the sequence $(|d_n|)_{n=1}^{\infty}$ is decreasing, we have that $(d_n)_{n=1}^{\infty} \in d(q)$ if and only if $(d_n)_{n=1}^{\infty} \in \ell^q$, and in this case the norms coincide. Consequently, $(n^{-s_0})_{n=1}^{\infty} \in \ell^q$ and, for $s_0 = \sigma + it$,

$$\|(n^{-s_0})_{n=1}^{\infty}\|_{d(q)} = \|(n^{-s_0})_{n=1}^{\infty}\|_{\ell^q} = \left(\sum_{n=1}^{\infty} \frac{1}{n^{\sigma q}}\right)^{1/q} = \zeta(\sigma q)^{1/q}.$$

In order to prove that $\|\delta_{s_0}\|$ becomes constant when $\sigma = \Re(s_0)$ is sufficiently large (only depending on p), we require the isometric identification of ces_p^* proved by Jagers, Theorem 2.5. Namely, for $(u_n)_{n=1}^{\infty} \in ces_p^*$ we have

$$\|(u_n)_{n=1}^{\infty}\|_{ces_p^*} = \left(\sum_{n\in D(u)} \left(\frac{|u_{m(n)}| - |u_{m(n+1)}|}{B_{m(n)} - B_{m(n+1)}}\right)^q (B_{m(n)} - B_{m(n+1)})\right)^{1/q}. \tag{2.6}$$

For $(n^{-s_0})_{n=1}^{\infty} \in ces_p^*$, with $s_0 \in \mathbb{C}_{1/q}$, we have that

$$m(1) = \max\{k \in \mathbb{N} \cup \{\infty\} : |k^{-s_0}| = \max_{j\geq 1} |j^{-s_0}|\} = 1,$$

since the sequence $(|n^{-s_0}|)_{n=1}^{\infty}$ is decreasing.

We claim that $m(2) = \infty$ provided that

$$\Re(s_0) = \sigma \geq \sigma_p := p - 1 + \frac{\log(p-1) + \log \zeta(p)}{\log 2}.$$

The condition $m(2) = \infty$ will follow if we prove, for every $n \geq 2$, that

$$\frac{|u_1| - |u_n|}{B_1 - B_n} \geq \frac{|u_1| - |u_\infty|}{B_1 - B_\infty} = \frac{|u_1|}{B_1}$$

This condition, for the sequence $(n^{-s_0})_{n=1}^\infty$, is

$$\frac{1 - \frac{1}{n^\sigma}}{\sum_{j=1}^{n-1} \frac{1}{j^p}} \geq \frac{1}{\sum_{j=1}^{\infty} \frac{1}{j^p}},$$

which is equivalent to

$$\sum_{j=n}^{\infty} \frac{1}{j^p} \geq \frac{1}{n^\sigma} \zeta(p).$$

Since

$$\sum_{j=n}^{\infty} \frac{1}{j^p} \geq \frac{1}{p-1} \cdot \frac{1}{n^{p-1}},$$

it suffices to prove that

$$\frac{1}{p-1} \cdot \frac{1}{n^{p-1}} \geq \frac{1}{n^\sigma} \zeta(p)$$

holds for all $n \geq 2$. We rewrite this condition as

$$n^{\sigma - p + 1} \geq (p-1)\zeta(p).$$

It is clear that for this last inequality to hold, necessarily we must have $\sigma \geq p - 1$. In this case, the sequence $(n^{\sigma-p+1})_{n=1}^\infty$ is increasing. Thus, it suffices to check the above inequality for $n = 2$:

$$2^{\sigma - p + 1} \geq (p-1)\zeta(p),$$

that is,

$$\sigma \geq p - 1 + \frac{\log(p-1) + \log \zeta(p)}{\log 2}.$$

Under this condition we have $m(2) = \infty$ for points $s_0 \in \mathbb{C}_{\sigma_p}$. Therefore, $D(b) = \{1\}$ for $s_0 \in \mathbb{C}_{\sigma_p}$, and the sum in (2.6) has only one term, and so

$$\begin{aligned}
\|\delta_{s_0}\|^q &= \left(\frac{|u_1| - |u_\infty|}{B_1 - B_\infty}\right)^q (B_1 - B_\infty) \\
&= B_1^{1-q} \\
&= \left(\sum_{n=1}^{\infty} n^{-p}\right)^{1-q} \\
&= \zeta(p)^{1-q}.
\end{aligned}$$

Consequently, $\|\delta_{s_0}\| = \zeta(p)^{(1-q)/q} = \zeta(p)^{-1/p}$ whenever $s \in \mathbb{C}_{\sigma_p}$. $\square$

Remark 2.14. From Theorem 2.13 and (2.5) we have, for $s_0 = \sigma + it \in \mathbb{C}_{1/q}$, that

$$\|\delta_{s_0}\| \leq \min\{\sigma, (p-1)^{1/p}\}\zeta(\sigma q)^{1/q}.$$

Since $1/q < (p-1)^{1/p}$ (as the function $x \mapsto x^x$ is increasing on $(1, \infty)$), we have that

$$\min\{\sigma, (p-1)^{1/p}\} = \begin{cases} \sigma & \text{for} \quad 1/q < \sigma \leq (p-1)^{1/p}, \\ (1-p)^{1/p} & \text{for} \quad \sigma > (p-1)^{1/p}. \end{cases}.$$

The bounds on the norm of point evaluations in Theorem 2.13 and Remark 2.14 can be sharpened for $\mathcal{H}(ces_2)$.

Proposition 2.15. *Let $1/2 < \Re(s_0) = \sigma \leq 1$ and $\delta_{s_0} \colon \mathcal{H}(ces_2) \to \mathbb{C}$ be the corresponding point evaluation functional. Then its norm can be written as*

$$\|\delta_{s_0}\| = \left(\sum_{n=1}^{\infty} n^2 \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right)^2 \right)^{1/2},$$

and the following bounds hold

$$(2^\sigma - 1)\sqrt{\zeta(2\sigma) - 1} \leq \|\delta_{s_0}\| \leq \sigma\sqrt{\zeta(2\sigma) - 1}.$$

Proof. We use the isometric identification of ces_p^* by Jagers in Theorem 2.5 for $p = 2$.

Let $u = (n^{-s_0})_{n=1}^{\infty}$. We will prove that in this case, and for every $m \in \mathbb{N}$, the sequence

$$\left(\frac{|u_m| - |u_n|}{B_m - B_n} \right)_{n=m+1}^{\infty} \tag{2.7}$$

is strictly increasing. This condition is precisely

$$\frac{\frac{1}{m^\sigma} - \frac{1}{n^\sigma}}{\sum_{k=m}^{\infty} \frac{1}{k^2} - \sum_{k=n}^{\infty} \frac{1}{k^2}} < \frac{\frac{1}{m^\sigma} - \frac{1}{(n+1)^\sigma}}{\sum_{k=m}^{\infty} \frac{1}{k^2} - \sum_{k=n+1}^{\infty} \frac{1}{k^2}}, \tag{2.8}$$

which can be written as

$$\frac{A}{B} < \frac{A+a}{B+b}, \tag{2.9}$$

for

$$\begin{aligned}
A &:= \frac{1}{m^\sigma} - \frac{1}{n^\sigma}, \\
a &:= \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}, \\
B &:= \sum_{k=m}^{\infty} \frac{1}{k^2} - \sum_{k=n}^{\infty} \frac{1}{k^2} = \sum_{k=m}^{n-1} \frac{1}{k^2}, \\
b &:= \frac{1}{n^2}.
\end{aligned}$$

The inequality (2.9) is equivalent to

$$\frac{Ab}{a} < B, \tag{2.10}$$

which is

$$\frac{\frac{1}{m^\sigma} - \frac{1}{n^\sigma}}{n^2\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right)} < \sum_{k=m}^{n-1} \frac{1}{k^2}. \tag{2.11}$$

We look at the left-hand side of (2.11). By applying the mean value theorem we obtain some real values $z_{m,n} \in (m,n)$ and $z_{n,n+1} \in (n, n+1)$, such that

$$\begin{aligned}
\frac{\frac{1}{m^\sigma} - \frac{1}{n^\sigma}}{n^2\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right)} &= \frac{n^\sigma (n+1)^\sigma (n^\sigma - m^\sigma)}{n^2 m^\sigma n^\sigma \left((n+1)^\sigma - n^\sigma\right)} \\
&= \frac{n^\sigma (n+1)^\sigma \sigma (n-m) z_{m,n}^{\sigma-1}}{n^2 m^\sigma n^\sigma \sigma z_{n,n+1}^{\sigma-1}}.
\end{aligned}$$

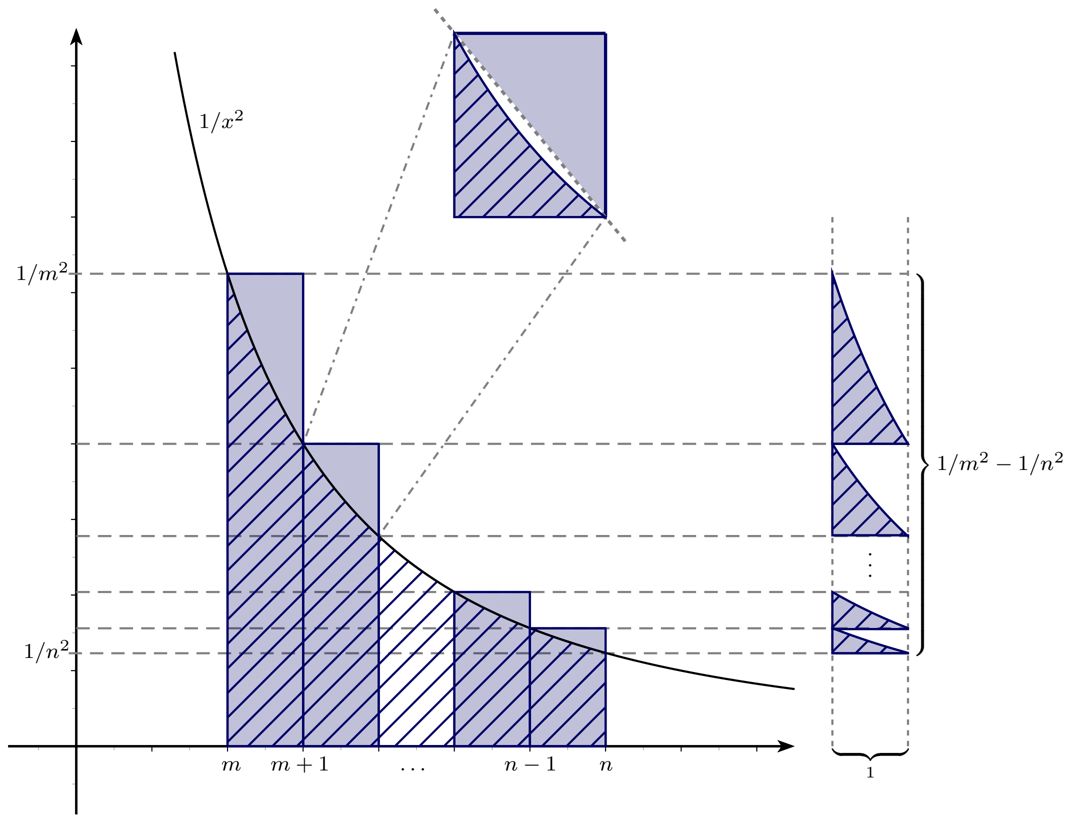
Then, for $1/2 < \sigma \leq 1$, we have

$$\begin{aligned}
\frac{Ab}{a} &= \frac{(n+1)^\sigma}{n^2 m^\sigma}(n-m)\frac{z_{m,n}^{\sigma-1}}{z_{n,n+1}^{\sigma-1}} \\
&\leq \frac{(n+1)^\sigma}{n^2 m^\sigma}(n-m)\frac{m^{\sigma-1}}{(n+1)^{\sigma-1}} \\
&= \frac{n+1}{n^2 m}(n-m). \qquad (2.12)
\end{aligned}$$

In order to bound the right-hand side of (2.11), that is, B, observe that

$$\int_m^n \frac{dx}{x^2} + \frac{1}{2}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \leq B = \sum_{k=m}^{n-1} \frac{1}{k^2}, \tag{2.13}$$

as shown in the next figure:



An analytic proof of (2.13) is the following: since the function $g(x) = 1/x^2$ is convex, the inequality

$$g\big((1-t)k + t(k+1)\big) \leq (1-t)g(k) + tg(k+1)$$

holds for any $t \in [0,1]$ and every k. This is equivalent to

$$g(x) \leq \big(g(k+1) - g(k)\big)(x-k) + g(k)$$

for $k \le x \le k+1$. Then

$$\begin{aligned}
\int_m^n \frac{dx}{x^2} &= \sum_{k=m}^{n-1} \int_k^{k+1} g(x)dx \\
&\le \sum_{k=m}^{n-1} \int_k^{k+1} \Big(\big(g(k+1)-g(k)\big)(x-k)+g(k)\Big)dx \\
&= \sum_{k=m}^{n-1} \frac{1}{2}\big(g(k+1)+g(k)\big) \\
&= \frac{1}{2}\sum_{k=m+1}^{n} g(k) + \frac{1}{2}\sum_{k=m}^{n-1} g(k) \\
&= \sum_{k=m+1}^{n-1} g(k) + \frac{1}{2}\big(g(n)+g(m)\big) \\
&= \sum_{k=m+1}^{n-1} \frac{1}{k^2} + \frac{1}{2}\left(\frac{1}{n^2}+\frac{1}{m^2}\right) = \sum_{k=m}^{n-1} \frac{1}{k^2} + \frac{1}{2}\left(\frac{1}{n^2}-\frac{1}{m^2}\right).
\end{aligned}$$

So, (2.13) holds.

On the other hand, it is also true that

$$\int_m^n \frac{dx}{x^2} + \frac{1}{2}\left(\frac{1}{m^2}-\frac{1}{n^2}\right) = \left(\frac{1}{m}-\frac{1}{n}\right)\left(1+\frac{1}{2m}+\frac{1}{2n}\right),$$

so

$$\left(\frac{n-m}{mn}\right)\left(1+\frac{1}{2m}+\frac{1}{2n}\right) \le B.$$

Then, by (2.12), we can prove the validity of (2.10) if we show

$$\frac{n+1}{n^2 m}(n-m) < \left(\frac{n-m}{mn}\right)\left(1+\frac{1}{2m}+\frac{1}{2n}\right).$$

But this last inequality can be rewritten as

$$\frac{n+1}{n} < 1+\frac{1}{2m}+\frac{1}{2n},$$

which is true since $m < n$. Thus, (2.8) holds and so, for every $m \in \mathbb{N}$, the sequence (2.7) is strictly increasing.

Thus, for each $n \in \mathbb{N}$ we have $m(n) = n$. So, Theorem 2.5 implies that $D(u) = \mathbb{N}$ for $u = (n^{-s_0})_{n=1}^{\infty}$ and so

$$\|\delta_{s_0}\| = \left\|(n^{-s_0})\right\|_{d(2)} = \left(\sum_{n=1}^{\infty} n^2\left(\frac{1}{n^\sigma}-\frac{1}{(n+1)^\sigma}\right)^2\right)^{1/2}. \tag{2.14}$$

Next we estimate each of the terms in the previous sum. For this we write them as follows:

$$\begin{aligned} n\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right) &= n \cdot \frac{(n+1)^\sigma - n^\sigma}{n^\sigma (n+1)^\sigma} \\ &= \frac{n}{(n+1)^\sigma} \cdot \left(\left(1+\frac{1}{n}\right)^\sigma - 1\right) \\ &= \frac{1}{(n+1)^\sigma} \cdot \frac{(1+n^{-1})^\sigma - 1}{n^{-1}} \\ &= \frac{1}{(n+1)^\sigma} \cdot \psi(n^{-1}), \end{aligned}$$

where $\psi(x) = \dfrac{(1+x)^\sigma - 1}{x}$. Simple calculations show that $\psi'(x) < 0$ for $x > 0$ and so ψ decreases in $(0, \infty)$, and

$$\psi(0^+) := \lim_{x \to 0^+} \psi(x) = \sigma.$$

Then,

$$2^\sigma - 1 = \psi(1) \leq \psi(n^{-1}) \leq \psi(0^+) = \sigma$$

and so

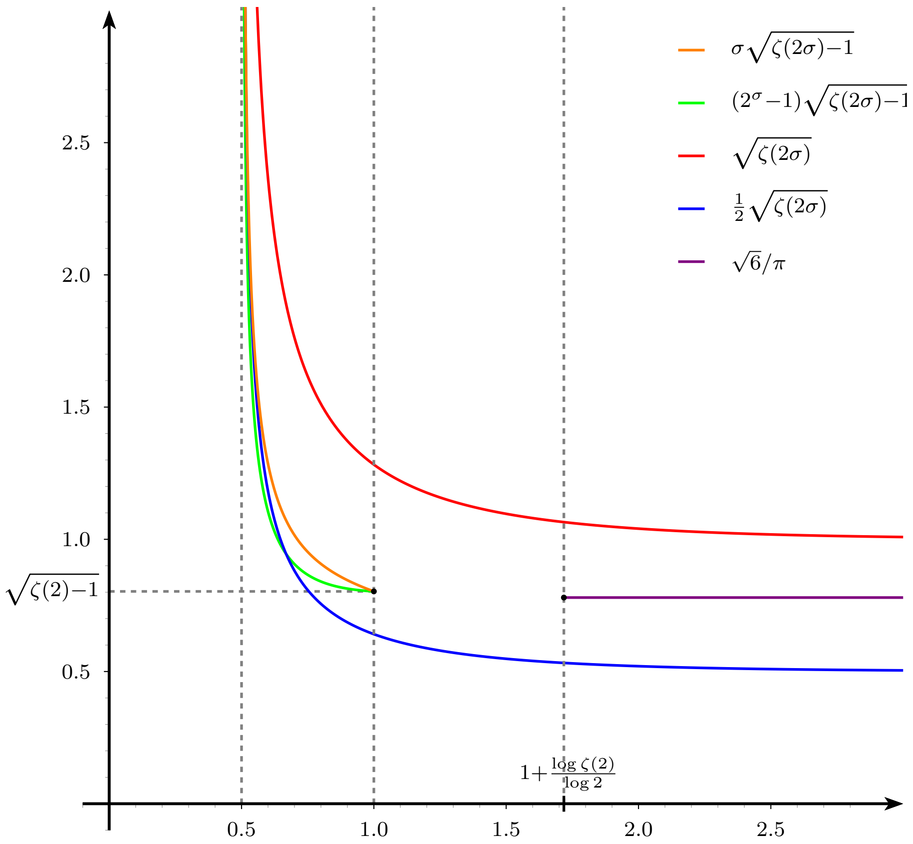
$$\frac{2^\sigma - 1}{(n+1)^\sigma} \leq n\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right) \leq \frac{\sigma}{(n+1)^\sigma}.$$

Finally make the corresponding sums over all $n \in \mathbb{N}$ to obtain.

$$(2^\sigma - 1)\sqrt{\zeta(2\sigma) - 1} \leq \|\delta_{s_0}\| \leq \sigma\sqrt{\zeta(2\sigma) - 1}.$$

$\square$

The following graph represents the bounds we have found for $\|\delta_{s_0}\|$, depending on $\Re(s_0)$, for point evaluations δ_{s_0} on $\mathcal{H}(ces_2)$.



We end this section exhibiting two equivalent expressions for the norm in $\mathcal{H}(ces_2)$ which are of independent interest.

Proposition 2.16. *Let* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}(ces_2)$. *Consider the functionals*

$$M(f) := \left(\sum_{i,j=1}^{\infty} \frac{|a_i||a_j|}{\max\{i,j\}} \right)^{1/2},$$

$$N(f) := \left(\sum_{n=1}^{\infty} \frac{|a_n|}{n} \sum_{k=1}^{n} |a_k| \right)^{1/2}.$$

Then

$$N(f) \leq M(f) \leq \|f\|_{\mathcal{H}(ces_2)} \leq \sqrt{2} M(f) \leq 2N(f).$$

Proof. Note that $(a_n)_{n=1}^\infty \in ces_2$. Then

$$\begin{aligned}
\|f\|^2_{\mathcal{H}(ces_2)} &= \|(a_n)_{n=1}^\infty\|^2_{ces_2} \\
&= \sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=1}^n |a_k|\right)^2 \\
&= \sum_{n=1}^\infty \frac{1}{n^2}\Bigg(\sum_{1\le i,j\le n} |a_i||a_j|\Bigg) \\
&= \sum_{i,j=1}^\infty |a_i||a_j|\Bigg(\sum_{n\ge i,j}\frac{1}{n^2}\Bigg).
\end{aligned}$$

Since $1/n \le \sum_{k=n}^\infty k^{-2} \le 2/n$, for every $n \ge 1$, it follows that

$$\frac{1}{\max\{i,j\}} \le \sum_{n\ge i,j}\frac{1}{n^2} \le \frac{2}{\max\{i,j\}}.$$

Hence, we deduce that

$$M(f) \le \|f\|_{\mathcal{H}(ces_2)} \le \sqrt{2}M(f).$$

On the other hand,

$$\begin{aligned}
M(f)^2 &= \sum_{i,j=1}^\infty \frac{|a_i||a_j|}{\max\{i,j\}} \\
&= \sum_{n=1}^\infty \frac{1}{n}\Bigg(\sum_{\max\{i,j\}=n} |a_i||a_j|\Bigg) \\
&= \sum_{n=1}^\infty \frac{|a_n|}{n}\Bigg(|a_n| + 2\sum_{k=1}^{n-1}|a_k|\Bigg) \\
&\le 2\sum_{n=1}^\infty \frac{|a_n|}{n}\sum_{k=1}^n |a_k| = 2N(f)^2.
\end{aligned}$$

In a similar way

$$M(f)^2 \ge \sum_{n=1}^\infty \frac{|a_n|}{n}\sum_{k=1}^n |a_k| = N(f)^2.$$

Consequently,

$$N(f) \le M(f) \le \sqrt{2}N(f).$$

$\square$

2.3 The Banach spaces of Dirichlet series $\mathcal{A}^r$, $\mathcal{A}^{2,r}$ and $\mathcal{H}^\infty(\mathbb{C}_r)$

There are other Banach spaces of Dirichlet series which will be needed in order to study the space $\mathcal{H}(ces_p)$. The most important ones are the spaces $\mathcal{A}^r$, $\mathcal{A}^{2,r}$, and $\mathcal{H}^\infty(\mathbb{C}_r)$ for $r \in \mathbb{R}$. We review them.

For $r \in \mathbb{R}$, the space $\mathcal{A}^r$ consists of all Dirichlet series whose sequence of coefficients belongs to the weighted space $\ell^1((n^{-r})_{n=1}^\infty)$, that is,

$$\mathcal{A}^r := \left\{ f(s) = \sum_{n=1}^\infty a_n n^{-s} : \sum_{n=1}^\infty |a_n| n^{-r} < \infty \right\}. \tag{2.15}$$

It is a solid Banach space under the coefficient-wise order and the norm

$$\|f\|_{\mathcal{A}^r} := \sum_{n=1}^\infty |a_n| n^{-r}, \quad f \in \mathcal{A}^r.$$

When $r = 0$, the corresponding space is the well known *Wiener-Dirichlet algebra* $\mathcal{A}^+$ consisting of all Dirichlet series with absolutely summable coefficients, studied by Bayart, Finet, Li, and H. Queffélec, see [13].

Proposition 2.17. *The following statements hold:*

(a) Regarding the abscissae of convergence we have

$$\sigma_c(\mathcal{A}^r) = \sigma_a(\mathcal{A}^r) = r.$$

(b) For $s_0 \in \overline{\mathbb{C}}_r$, the point evaluation functional δ_{s_0} is bounded on $\mathcal{A}^r$ and $\|\delta_{s_0}\| = 1$.

Proof. (a) Note that $\sigma_a(f) \le r$ for all $f \in \mathcal{A}^r$ and thus $\sigma_a(\mathcal{A}^r) \le r$. For arbitrary $\alpha > 1 - r$, we have that $f_\alpha(s) := \sum_{n=1}^\infty n^{-\alpha} n^{-s} \in \mathcal{A}^r$ and

$$\sigma_c(\mathcal{A}^r) \ge \sigma_c(f_\alpha) = 1 - \alpha.$$

Making $\alpha \to 1 - r$ it follows that $\sigma_c(\mathcal{A}^r) \ge r$ and so the conclusion follows.

(b) For $s_0 \in \overline{\mathbb{C}}_r$ and $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{A}^r$, we have that

$$|\delta_{s_0}(f)| = |f(s_0)| \le \sum_{n=1}^\infty \frac{|a_n|}{n^{\Re(s_0)}} \le \sum_{n=1}^\infty \frac{|a_n|}{n^r} = \|f\|_{\mathcal{A}^r}.$$

Hence $\delta_{s_0} \colon \mathcal{A}^r \to \mathbb{C}$ is continuous with $\|\delta_{s_0}\| \le 1$. Actually $\|\delta_{s_0}\| = 1$ as the function $\mathbf{1}$ belongs to $\mathcal{A}^r$ and has norm equal to one. $\square$

In a similar way, for $r \in \mathbb{R}$, the space $\mathcal{A}^{2,r}$ consists of all Dirichlet series whose sequence of coefficients belongs to the weighted space $\ell^2((n^{-2r})_{n=1}^{\infty})$, that is,

$$\mathcal{A}^{2,r} := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=1}^{\infty} |a_n|^2 n^{-2r} < \infty \right\}. \tag{2.16}$$

It is a solid Banach space under the coefficient-wise order and the norm

$$\|f\|_{\mathcal{A}^{2,r}} := \left(\sum_{n=1}^{\infty} |a_n|^2 n^{-2r} \right)^{1/2}, \quad f \in \mathcal{A}^{2,r}.$$

When $r = 0$, the corresponding space is the *Hilbert space of Dirichlet series* $\mathcal{H}$ consisting of all Dirichlet series with square summable coefficients, studied by Hedenmalm, Lindqvist, and K. Seip, see [35].

Proposition 2.18. *The following statements hold:*

(a) Regarding the abscissae of convergence we have

$$\sigma_c(\mathcal{A}^{2,r}) = \sigma_a(\mathcal{A}^{2,r}) = 1/2 + r.$$

(b) For $s_0 = \sigma + it \in \mathbb{C}_{1/2+r}$, the point evaluation functional δ_{s_0} is bounded on $\mathcal{A}^{2,r}$ and

$$\|\delta_{s_0}\| \leq \sqrt{\zeta(2(\sigma - r))}.$$

Proof. (a) Note, for every $f \in \mathcal{A}^{2,r}$ and $\sigma > 1/2 + r$, that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} &\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^{2(\sigma-r)}} \right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2r}} \right)^{1/2} \\ &= \sqrt{\zeta(2(\sigma - r))}\, \|f\|_{\mathcal{A}^{2,r}}, \end{aligned} \tag{2.17}$$

and so $\sigma_a(f) \leq 1/2 + r$. Hence, $\sigma_a(\mathcal{A}^{2,r}) \leq 1/2 + r$. For arbitrary $\alpha > 1/2 - r$, we have that $f_\alpha(s) := \sum_{n=1}^{\infty} n^{-\alpha} n^{-s} \in \mathcal{A}^{2,r}$ and

$$\sigma_c(\mathcal{A}^{2,r}) \geq \sigma_c(f_\alpha) = 1 - \alpha.$$

Making $\alpha \to 1/2 - r$ it follows that $\sigma_c(\mathcal{A}^{2,r}) \geq 1/2 + r$. The result is thus established.

(b) For each $s_0 = \sigma + it \in \mathbb{C}_{1/2+r}$ and $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{A}^{2,r}$, by (2.17) we have that

$$|\delta_{s_0}(f)| = |f(s_0)| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} \leq \sqrt{\zeta(2(\sigma - r))}\, \|f\|_{\mathcal{A}^{2,r}}.$$

Then $\delta_{s_0} \colon \mathcal{A}^{2,r} \to \mathbb{C}$ is continuous with $\|\delta_{s_0}\| \leq \sqrt{\zeta(2(\sigma - r))}$. $\square$

Remark 2.19. The following continuous embeddings hold

$$\mathcal{A}^{1/2} \subseteq \mathcal{H}(ces_2) \subseteq \mathcal{A}^{2,1/2}$$

Indeed, for $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, since

$$|a_n| \leq \sum_{k=1}^{n} |a_k| \leq \sqrt{n} \sum_{k=1}^{n} |a_k| k^{-1/2},$$

considering the equivalent expression $N(f)$ for the norm in $\mathcal{H}(ces_2)$ given in Proposition 2.16 we have that

$$\sum_{n=1}^{\infty} \frac{|a_n|^2}{n} \leq N(f)^2 \leq \sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{n}} \sum_{k=1}^{n} \frac{|a_k|}{\sqrt{k}} \leq \left(\sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{n}} \right)^2.$$

For $r \in \mathbb{R}$, the space $\mathcal{H}^\infty(\mathbb{C}_r)$ consists of all bounded analytic functions on $\mathbb{C}_r$ which can be represented as a Dirichlet series in some vertical half-plane (actually in $\mathbb{C}_r$ by Bohr's Theorem 1.15), that is,

$$\mathcal{H}^\infty(\mathbb{C}_r) = \mathcal{D} \cap H^\infty(\mathbb{C}_r). \tag{2.18}$$

It is a linear space that will be endowed with the supremum norm

$$\|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)} := \sup_{s \in \mathbb{C}_r} |f(s)|, \quad f \in \mathcal{H}^\infty(\mathbb{C}_r).$$

Note that these spaces are not solid for the coefficient-wise order.

Proposition 2.20. *The following statements hold:*

(a) Regarding the abscissae of convergence we have

$$\sigma_c(\mathcal{H}^\infty(\mathbb{C}_r)) = r \quad \text{and} \quad \sigma_a(\mathcal{H}^\infty(\mathbb{C}_r)) = r + 1/2.$$

(b) For each $s_0 \in \mathbb{C}_r$, the point evaluation functional δ_{s_0} is bounded on $\mathcal{H}^\infty(\mathbb{C}_r)$ and $\|\delta_{s_0}\| = 1$.

Proof. (a) As noted before if $f \in \mathcal{H}^\infty(\mathbb{C}_r)$ then $\sigma_c(f) \leq r$ and so $\sigma_c(\mathcal{H}^\infty(\mathbb{C}_r)) \leq r$. For arbitrary $\alpha > 1 - r$, we have that $f_\alpha(s) := \sum_{n=1}^{\infty} n^{-\alpha} n^{-s} \in \mathcal{H}^\infty(\mathbb{C}_r)$ and

$$\sigma_c(\mathcal{H}^\infty(\mathbb{C}_r)) \geq \sigma_c(f_\alpha) = 1 - \alpha.$$

Making $\alpha \to 1 - r$ it follows that $\sigma_c(\mathcal{H}^\infty(\mathbb{C}_r)) \geq r$. Thus, $\sigma_c(\mathcal{H}^\infty(\mathbb{C}_r)) = r$.

For the abscissa of absolute convergence, we first consider the case $r = 0$. It follows from results of Bohnenblust and Hille, and Balasubramanian, Calado and Queffélec, see [9, Theorem 1.1.2)], that

(i) $\sum_{n=1}^{\infty} |a_n| n^{-1/2} < \infty$ for all $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty(\mathbb{C}_0)$.

(ii) For each $0 \le \alpha < 1/2$ there exists $f_\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty(\mathbb{C}_0)$ such that $\sum_{n=1}^{\infty} |a_n| n^{-\alpha} = \infty$.

These facts imply that $\sigma_a\big(\mathcal{H}^\infty(\mathbb{C}_0)\big) = 1/2$. For $r \neq 0$ consider the *translation map* $\tau_r : \mathcal{D} \to \mathcal{D}$ given by

$$\tau_r(f)(s) := f(s+r), \tag{2.19}$$

that is,

$$\tau_r\left(\sum_{n=1}^{\infty} a_n n^{-s}\right) = \sum_{n=1}^{\infty} a_n n^{-(s+r)} = \sum_{n=1}^{\infty} (a_n n^{-r}) n^{-s}.$$

Note that $\sigma_c(\tau_r(f)) = \sigma_c(f) - r$ and the same holds for the abscissae of absolute and uniform convergence. The translation τ_r establishes an isometric isomorphism between $\mathcal{H}^\infty(\mathbb{C}_r)$ and $\mathcal{H}^\infty(\mathbb{C}_0)$,

$$\tau_r : \mathcal{H}^\infty(\mathbb{C}_r) \to \mathcal{H}^\infty(\mathbb{C}_0),$$

from which it follows that

$$\sigma_a(\mathcal{H}^\infty(\mathbb{C}_r)) = r + \sigma_a(\mathcal{H}^\infty(\mathbb{C}_0)) = r + 1/2.$$

Part (b) is obvious. □

The isometric isomorphism between $\mathcal{H}^\infty(\mathbb{C}_r)$ and $\mathcal{H}^\infty(\mathbb{C}_0)$ allows a neat solution to establishing the completeness of $\mathcal{H}^\infty(\mathbb{C}_r)$ for the supremum norm. The result of Hedenmalm, Lindqvist and Seip states that $\mathcal{H}^\infty(\mathbb{C}_0)$ is isometrically isomorphic to the $\mathcal{M}(\mathcal{H})$, the multiplier algebra of the Hilbert space of Dirichlet series $\mathcal{H}$ (Theorem 1.4). Since this last space is complete (for the operator norm) it follows that $\mathcal{H}^\infty(\mathbb{C}_0)$ is complete for the supremum norm. Hence, $\mathcal{H}^\infty(\mathbb{C}_r)$ endowed with the supremum norm is a Banach space.

The relation between the spaces $\mathcal{A}^r$ and $\mathcal{H}^\infty(\mathbb{C}_r)$ is clear:

$$\mathcal{A}^r \subseteq \mathcal{H}^\infty(\mathbb{C}_r) \tag{2.20}$$

and the containment is continuous with embedding constant equal to one. Indeed, for $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathcal{A}^r$ and $s = \sigma + it \in \mathbb{C}_r$, we have

$$|f(s)| \le \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \le \sum_{n=1}^{\infty} |a_n| n^{-r} = \|f\|_{\mathcal{A}^r}.$$

However, that relation is in fact more precise.

Proposition 2.21. *Let $f \in \mathcal{H}^\infty(\mathbb{C}_r)$ with $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ such that $a_n \geq 0$ for all $n \geq 1$. Then $f \in \mathcal{A}^r$ and*

$$\|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)} = \|f\|_{\mathcal{A}^r}.$$

Thus, the positive cones of $\mathcal{A}^r$ and $\mathcal{H}^\infty(\mathbb{C}_r)$ coincide. Moreover, the space $\mathcal{A}^r$ is the solid core of $\mathcal{H}^\infty(\mathbb{C}_r)$.

Proof. For $\alpha > r$ and $N \in \mathbb{N}$, as $a_n \geq 0$ for all $n \geq 1$, we have

$$\|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)} \geq |f(\alpha)| = \left| \sum_{n=1}^\infty a_n n^{-\alpha} \right| \geq \sum_{n=1}^N a_n n^{-\alpha}.$$

Making $\alpha \to r$ it follows that $\sum_{n=1}^N a_n n^{-r} \leq \|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)}$. So, $f \in \mathcal{A}^r$, and $\|f\|_{\mathcal{A}^r} \leq \|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)}$. Thus, both norms coincide. This shows that the positive cones of $\mathcal{A}^r$ and $\mathcal{H}^\infty(\mathbb{C}_r)$ coincide.

In order to show that $\mathcal{A}^r$ is the solid core of $\mathcal{H}^\infty(\mathbb{C}_r)$, let $\mathcal{E}$ be a solid subspace of $\mathcal{H}^\infty(\mathbb{C}_r)$. We have to show that $\mathcal{E} \subseteq \mathcal{A}^r$. Let $f \in \mathcal{E}$ with $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ and denote by h the Dirichlet series defined as $h(s) := \sum_{n=1}^\infty |a_n| n^{-s}$. Since $\mathcal{E}$ is solid, we have that $h \in \mathcal{E} \subseteq \mathcal{H}^\infty(\mathbb{C}_r)$. Then $h \in \mathcal{A}^r$ and so $f \in \mathcal{A}^r$, as $\mathcal{A}^r$ is solid. Consequently, $\mathcal{E} \subseteq \mathcal{A}^r$. □

Not only Dirichlet series in $\mathcal{H}^\infty(\mathbb{C}_r)$ with nonnegative coefficients belong to $\mathcal{A}^r$, but also some other series with real coefficients, that is, $\sum_{n=1}^\infty a_n n^{-s}$ in $\mathcal{H}^\infty(\mathbb{C}_r)$ such that $a_n \in \mathbb{R}$, for all $n \geq 1$, whenever the sign of the coefficients behaves multiplicatively. Recall that sgn denotes the sign function.

Theorem 2.22. *Let $f \in \mathcal{H}^\infty(\mathbb{C}_r)$ with $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ such that $a_n \in \mathbb{R}$ for all $n \geq 1$, and*

$$\operatorname{sgn}(a_{ij}) = \operatorname{sgn}(a_i) \cdot \operatorname{sgn}(a_j), \quad i, j \geq 1. \tag{2.21}$$

Then $f \in \mathcal{A}^r$, and, moreover,

$$\|f\|_{\mathcal{A}^r} = \|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)}.$$

Proof. Recall that the translation operator τ_r, which maps f into $f(\cdot + r)$, establishes an isometric isomorphism between the spaces $\mathcal{H}^\infty(\mathbb{C}_r)$ and $\mathcal{H}^\infty(\mathbb{C}_0)$. The operator τ_r is also an isometric isomorphism between $\mathcal{A}^r$ and $\mathcal{A}^0$. This fact, together with

$$\operatorname{sgn}(a_n n^{-r}) = \operatorname{sgn}(a_n), \quad n \geq 1,$$

shows that in order to prove the result it suffices to establish the following: if $f \in \mathcal{H}^\infty = \mathcal{H}^\infty(\mathbb{C}_0)$ satisfies condition (2.21), then $f \in \mathcal{A}^0$ and $\|f\|_{\mathcal{A}^0} = \|f\|_{\mathcal{H}^\infty}$.

The result of Hedenmalm, Lindqvist and Seip states that $\mathcal{M}(\mathcal{H}) = \mathcal{H}^\infty$ (Theorem 1.4). Thus, we can assume that $f \in \mathcal{M}(\mathcal{H})$. We can also assume that $a_n \neq 0$ for some $n \geq 1$, as in other case $f \equiv 0$ and there would be nothing to prove. Fix $M \in \mathbb{N}$ such that $a_M \neq 0$.

Choose $m \in \mathbb{N}$, and set $b := (b_n)_{n=1}^\infty$ with

$$b_n = \begin{cases} \operatorname{sgn}(a_n) & \text{if } n|M^m \\ 0 & \text{in other case.} \end{cases}$$

Note that if $n|M^m$ then $b_n \neq 0$, as

$$\operatorname{sgn}(a_n) \cdot \operatorname{sgn}(a_{M^m/n}) = \operatorname{sgn}(a_{M^m}) = \operatorname{sgn}(a_M)^m \neq 0.$$

Let $g(s) := \sum_{n=1}^\infty b_n n^{-s}$. Thus, $g \in \mathcal{H}$, since it has a finite number of non-zero coordinates.

Then, for $\operatorname{card}(I)$ denoting the cardinal of the set I, we have

$$\|fg\|_{\mathcal{H}} \leq \|f\|_{\mathcal{M}(\mathcal{H})} \|g\|_{\mathcal{H}} = \|f\|_{\mathcal{M}(\mathcal{H})} \big(\operatorname{card}(\{n : n|M^m\})\big)^{1/2}. \tag{2.22}$$

On the other hand,

$$\begin{aligned} \|fg\|_{\mathcal{H}}^2 &= \sum_{n=1}^\infty \Big| \sum_{k|n} b_k a_{\frac{n}{k}} \Big|^2 \\ &\geq \sum_{n \in \{jM : j|M^{m-1}\}} \Big| \sum_{k|n} b_k a_{\frac{n}{k}} \Big|^2 \\ &= \sum_{n \in \{jM : j|M^{m-1}\}} \Big| \sum_{k|n} \operatorname{sgn}(a_k) a_{\frac{n}{k}} \Big|^2, \end{aligned}$$

where for the last equality note that if k is a divisor of n with $n = jM$ and $j|M^{m-1}$, then k is a divisor of M^m, and so $b_k = \operatorname{sgn}(a_k)$. Next, appealing to the assumption on multiplicativity of the signs, we have

$$\begin{aligned}
\sum_{n\in\{jM:\,j|M^{m-1}\}} \left|\sum_{k|n} \operatorname{sgn}(a_k) a_{\frac{n}{k}}\right|^2 &= \sum_{n\in\{jM:\,j|M^{m-1}\}} \left|\sum_{k|n} \operatorname{sgn}(a_k)\operatorname{sgn}(a_{\frac{n}{k}})|a_{\frac{n}{k}}|\right|^2 \\
&= \sum_{n\in\{jM:\,j|M^{m-1}\}} \left|\sum_{k|n} \operatorname{sgn}(a_n)|a_{\frac{n}{k}}|\right|^2 \\
&= \sum_{n\in\{jM:\,j|M^{m-1}\}} \left(\sum_{k|n} |a_{\frac{n}{k}}|\right)^2 \\
&= \sum_{n\in\{jM:\,j|M^{m-1}\}} \left(\sum_{i|n} |a_i|\right)^2,
\end{aligned}$$

where the second to last equality holds since $a_n \neq 0$ for every $n = jM$ such that $j|M^{m-1}$, as in this case $n|M^m$. Since every i which is a divisor of M is a divisor of $n = jM$, it follows that

$$\begin{aligned}
\|fg\|_{\mathcal{H}}^2 &\geq \left(\sum_{i|M} |a_i|\right)^2 \operatorname{card}(\{jM : j|M^{m-1}\}) \\
&= \left(\sum_{i|M} |a_i|\right)^2 \operatorname{card}(\{j : j|M^{m-1}\}). \qquad (2.23)
\end{aligned}$$

From (2.22) and (2.23), we have

$$\sum_{i|M} |a_i| \leq \|f\|_{\mathcal{M}(\mathcal{H})} \left(\frac{\operatorname{card}(\{n : n|M^m\})}{\operatorname{card}(\{j : j|M^{m-1}\})}\right)^{1/2}.$$

Consider the prime factorization of M, that is, $M = \prod_{i=1}^{n_0} p_i^{\alpha_i}$. Then, $M^m = \prod_{i=1}^{n_0} p_i^{\alpha_i m}$ and $M^{m-1} = \prod_{i=1}^{n_0} p_i^{\alpha_i(m-1)}$, so

$$\begin{aligned}
\sum_{i|M} |a_i| &\leq \|f\|_{\mathcal{M}(\mathcal{H})} \left(\frac{\prod_{i=1}^{n_0}(\alpha_i m+1)}{\prod_{i=1}^{n_0}\left(\alpha_i(m-1)+1\right)}\right)^{1/2} \\
&= \|f\|_{\mathcal{M}(\mathcal{H})} \prod_{i=1}^{n_0} \left(\frac{\alpha_i m + 1}{\alpha_i(m-1)+1}\right)^{1/2}.
\end{aligned}$$

Taking limit as $m \to \infty$ yields

$$\sum_{i|M} |a_i| \leq \|f\|_{\mathcal{M}(\mathcal{H})}. \qquad (2.24)$$

Let $N \in \mathbb{N}$ be large enough so that the set $I_N = \{i : 1 \le i \le N,\, a_i \ne 0\}$ is not empty. Consider $M_N := \prod_{i \in I_N} i$. Note that $a_{M_N} \ne 0$ as $\operatorname{sgn}(a_{M_N}) = \prod_{i \in I_N} \operatorname{sgn}(a_i) \ne 0$. It follows from (2.24) that

$$\sum_{i=1}^{N} |a_i| = \sum_{i \in I_N} |a_i| \le \sum_{i | M_N} |a_i| \le \|f\|_{\mathcal{M}(\mathcal{H})}.$$

Therefore, $f \in \mathcal{A}^0$ and $\|f\|_{\mathcal{A}^0} \le \|f\|_{\mathcal{M}(\mathcal{H})} = \|f\|_{\mathcal{H}^\infty}$. Since we always have $\|f\|_{\mathcal{A}^0} \ge \|f\|_{\mathcal{H}^\infty}$, it follows that $\|f\|_{\mathcal{A}^0} = \|f\|_{\mathcal{H}^\infty}$. □

2.4 Other Banach spaces of Dirichlet series

We end this chapter by considering briefly other Banach spaces of Dirichlet series, related to the sequence space ces_p.

For $1 < p < \infty$, the space of Dirichlet series $\mathcal{H}(\ell^p)$ consists of all Dirichlet series whose sequence of coefficients belongs to the space ℓ^p, that is,

$$\mathcal{H}(\ell^p) := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}. \tag{2.25}$$

It is a solid Banach space under the coefficient-wise order and the norm

$$\|f\|_{\mathcal{H}(\ell^p)} := \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}, \quad f \in \mathcal{H}(\ell^p).$$

The case $p = 2$ corresponds to the Hilbert space of Dirichlet series $\mathcal{H}$.

Proposition 2.23. *For q being the conjugate exponent of p, the following statements hold:*

(a) Regarding the abscissae of convergence we have

$$\sigma_c(\mathcal{H}(\ell^p)) = \sigma_a(\mathcal{H}(\ell^p)) = 1/q.$$

(b) For $s_0 = \sigma + it \in \mathbb{C}_{1/q}$, the point evaluation functional δ_{s_0} is bounded on $\mathcal{H}(\ell^p)$ and

$$\|\delta_{s_0}\| = \zeta(\sigma q)^{1/q}.$$

Proof. (a) Note, for every $f \in \mathcal{H}(\ell^p)$ and $\sigma > 1/q$, that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \leq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{1}{n^{q\sigma}}\right)^{1/q} = \zeta(\sigma q)^{1/q} \|f\|_{\mathcal{H}(\ell^p)},$$

and so $\sigma_a(f) \leq 1/q$. Hence, $\sigma_a(\mathcal{H}(\ell^p)) \leq 1/q$. For arbitrary $\alpha > 1/p$ we have that $f_\alpha(s) := \sum_{n=1}^{\infty} n^{-\alpha} n^{-s} \in \mathcal{H}(\ell^p)$ and

$$\sigma_c(\mathcal{H}(\ell^p)) \geq \sigma_c(f_\alpha) = 1 - \alpha.$$

Making $\alpha \to 1/p$ it follows that $\sigma_c(\mathcal{H}(\ell^p)) \geq 1 - 1/p = 1/q$. The result is thus established.

(b) For $s_0 = \sigma + it \in \mathbb{C}_{1/q}$, since

$$\delta_{s_0}(f) = f(s_0) = \sum_{n=1}^{\infty} a_n n^{-s_0} = \left\langle \left(n^{-s_0}\right)_{n=1}^{\infty}, (a_n)_{n=1}^{\infty} \right\rangle_{\ell^p}, \quad f \in \mathcal{H}(\ell^p),$$

and $(n^{-s_0})_{n=1}^{\infty}$ belongs to ℓ^q, as $\sigma > 1/q$, we have that $\delta_{s_0} \colon \mathcal{H}(\ell^p) \to \mathbb{C}$ is continuous with $\|\delta_{s_0}\| = \|(n^{-s_0})_{n=1}^{\infty}\|_{\ell^q} = \zeta(\sigma q)^{1/q}$. $\square$

The next space of Dirichlet series involves the largest domain $[\mathcal{C}, \ell^p]$ of definition in $\mathbb{C}^{\mathbb{N}}$ for the Cesàro averaging operator $\mathcal{C}$ considered with values in ℓ^p, that is, for $1 < p < \infty$,

$$[\mathcal{C}, \ell^p] := \left\{ a = (a_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : \ \mathcal{C}(a) = \left(\frac{1}{n} \sum_{k=1}^{n} a_k\right)_{n=1}^{\infty} \in \ell^p \right\}.$$

It is a Banach space under the norm $\|a\|_{[\mathcal{C},\ell^p]} := \|\mathcal{C}(a)\|_{\ell^p}$ for $a \in [\mathcal{C}, \ell^p]$, which makes $\mathcal{C} \colon [\mathcal{C}, \ell^p] \to \ell^p$ an isometry. This space was studied by Curbera and Ricker in [25]. In particular, they noted that $[\mathcal{C}, \ell^p]$ is not solid for the coordinate-wise order, in fact its solid core is just the space ces_p.

The space of Dirichlet series $\mathcal{H}([\mathcal{C}, \ell^p])$ consists of all Dirichlet series whose sequence of coefficients belongs to the space $[\mathcal{C}, \ell^p]$, that is,

$$\mathcal{H}([\mathcal{C}, \ell^p]) := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \ \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right|^p < \infty \right\}. \tag{2.26}$$

It is a Banach space under the norm

$$\|f\|_{\mathcal{H}([\mathcal{C},\ell^p])} := \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right|^p \right)^{1/p}, \quad f \in \mathcal{H}([\mathcal{C}, \ell^p]).$$

Since $\ell^p \subsetneq ces_p \subsetneq [\mathcal{C}, \ell^p]$ it follows that

$$\mathcal{H}(\ell^p) \subsetneq \mathcal{H}(ces_p) \subsetneq \mathcal{H}([\mathcal{C}, \ell^p]).$$

Moreover $\mathcal{H}(ces_p)$ is the solid core of $\mathcal{H}([\mathcal{C}, \ell^p])$ for the coefficient-wise order.

Proposition 2.24. *For q being the conjugate exponent of p, the following statements hold:*

(a) Regarding the abscissae of convergence we have

$$\sigma_c(\mathcal{H}([\mathcal{C}, \ell^p])) = 1 \quad \text{and} \quad \sigma_a(\mathcal{H}([\mathcal{C}, \ell^p])) = 1 + 1/q.$$

(b) For each $s_0 = \sigma + it \in \mathbb{C}_{1+1/q}$, the point evaluation functional δ_{s_0} is bounded on $\mathcal{H}([\mathcal{C}, \ell^p])$ and

$$\|\delta_{s_0}\| \leq 2\zeta(q(\sigma - 1))^{1/q}.$$

Proof. (a) We first show that $\sigma_c(\mathcal{H}([\mathcal{C}, \ell^p])) \leq 1$. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}([\mathcal{C}, \ell^p])$. If $\sigma_c(f) \leq 0$ then $\sigma_c(f) \leq 1$. Suppose that $\sigma_c(f) > 0$. In particular $\sum_{n=1}^{\infty} a_n$ diverges and so we can use Cahen's formula (Theorem 1.14) to calculate the abscissa of convergence of f. Since $\mathcal{C}\big((a_n)_{n=1}^{\infty}\big) \in \ell^p$, we have that $\frac{1}{N}\big|\sum_{k=1}^{N} a_k\big| \leq 1$ for all N large enough, and so

$$\sigma_c(f) = \limsup_{N\to\infty} \frac{\log\big|\sum_{k=1}^{N} a_k\big|}{\log N} \leq \limsup_{N\to\infty} \frac{\log N}{\log N} = 1.$$

Let us see now that the value 1 cannot be improved. For each $0 < \alpha < 1$, let $b^\alpha = (b_n^\alpha)_{n=1}^{\infty} \in \ell^1 \subset \ell^p$ defined by

$$b_n^\alpha := \begin{cases} n^{\alpha-1}\log n & \text{if } n = 2^k \text{ for some } k \geq 1, \\ 0 & \text{in other case.} \end{cases}$$

Then, take $a^\alpha = (a_n^\alpha)_{n=1}^{\infty} \in [\mathcal{C}, \ell^p]$ given by

$$a^\alpha := \mathcal{C}^{-1}(b^\alpha) = \big(nb_n^\alpha - (n-1)b_{n-1}^\alpha\big)_{n=1}^{\infty},$$

where $b_0 = 0$ and $\mathcal{C}^{-1}$ is the inverse of the Cesàro operator. Thus, the Dirichlet series $f_\alpha(s) := \sum_{n=1}^{\infty} a_n^\alpha n^{-s}$ associated to a^α belongs to $\mathcal{H}([\mathcal{C}, \ell^p])$. Note that $\sum_{n=1}^{\infty} a_n^\alpha$ diverges, as

$$\begin{aligned} \sum_{n=1}^{N} a_n^\alpha &= \sum_{n=1}^{N} \big(nb_n^\alpha - (n-1)b_{n-1}^\alpha\big) \\ &= Nb_N^\alpha = \begin{cases} N^\alpha \log N & \text{if } N = 2^k \text{ for some } k \geq 1, \\ 0 & \text{in other case.} \end{cases} \end{aligned}$$

Applying Cahen's formula,

$$\begin{aligned}\sigma_c(f_\alpha) &= \limsup_{N\to\infty} \frac{\log\left|\sum_{n=1}^N a_n^\alpha\right|}{\log N}\\ &= \limsup_{N\to\infty} \frac{\log(Nb_N^\alpha)}{\log N}\\ &= \lim_{N\to\infty} \frac{\log\big(N^\alpha \log N\big)}{\log N}\\ &= \alpha + \lim_{N\to\infty} \frac{\log\big(\log N\big)}{\log N} = \alpha,\end{aligned}$$

and so $\sigma_c(\mathcal{H}([\mathcal{C},\ell^p])) \geq \sigma_c(f_\alpha) = \alpha$. Making $\alpha \to 1$, we obtain $\sigma_c(\mathcal{H}([\mathcal{C},\ell^p])) \geq 1$. Hence, $\sigma_c(\mathcal{H}([\mathcal{C},\ell^p])) = 1$.

For the abscissa of absolute convergence, we first show that $\sigma_a(\mathcal{H}([\mathcal{C},\ell^p])) \leq 1 + 1/q$. Let $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}([\mathcal{C},\ell^p])$. If $\sigma_a(f) \leq 0$ then $\sigma_a(f) \leq 1 + 1/q$. Suppose that $\sigma_a(f) > 0$, in which case $\sum_{n=1}^\infty |a_n|$ diverges. Then, Cahen's formula gives

$$\sigma_a(f) = \limsup_{N\to\infty} \frac{\log \sum_{n=1}^N |a_n|}{\log N}.$$

Note that $a_n = \sum_{k=1}^n a_k - \sum_{k=1}^{n-1} a_k$ for all $n \geq 2$. Thus,

$$\begin{aligned}\sum_{n=1}^N |a_n| &\leq |a_1| + \sum_{n=2}^N \left|\sum_{k=1}^n a_k\right| + \sum_{n=2}^N \left|\sum_{k=1}^{n-1} a_k\right|\\ &= \sum_{n=1}^N \left|\sum_{k=1}^n a_k\right| + \sum_{n=1}^{N-1} \left|\sum_{k=1}^n a_k\right|\\ &\leq 2\sum_{n=1}^N \left|\sum_{k=1}^n a_k\right|\\ &\leq 2N\sum_{n=1}^N \frac{1}{n}\left|\sum_{k=1}^n a_k\right|\end{aligned}$$

and so

$$\sum_{n=1}^N |a_n| \leq 2N\left(\sum_{n=1}^N \frac{1}{n^p}\left|\sum_{k=1}^n a_k\right|^p\right)^{1/p} N^{1/q},$$

that is,

$$\sum_{n=1}^N |a_n| \leq 2N^{1+1/q}\|f\|_{\mathcal{H}([\mathcal{C},\ell^p])}.$$

Therefore,

$$\begin{aligned}\sigma_a(f) &\leq \limsup_{N\to\infty} \frac{\log\left(2N^{1+1/q}\|f\|_{\mathcal{H}([\mathcal{C},\ell^p])}\right)}{\log N} \\ &= 1+1/q.\end{aligned}$$

Now, let us see that the value $1+1/q$ cannot be improved. For $0<\alpha$ consider the sequence $\left((-1)^n n^\alpha\right)_{n=1}^\infty$. Note that

$$\left|\sum_{k=1}^n (-1)^k k^\alpha\right| < n^\alpha, \quad n\geq 1.$$

Indeed, whenever n is even

$$\begin{aligned}\sum_{k=1}^n (-1)^k k^\alpha &= -1+\underbrace{2^\alpha-3^\alpha}_{<0}+\underbrace{4^\alpha-5^\alpha}_{<0}+\cdots+\underbrace{(n-2)^\alpha-(n-1)^\alpha}_{<0}+n^\alpha<n^\alpha \\ \sum_{k=1}^n (-1)^k k^\alpha &= \overbrace{-1+2^\alpha}^{>0}\;\overbrace{-3^\alpha+4^\alpha}^{>0}-5^\alpha+\cdots+(n-2)^\alpha\;\overbrace{-(n-1)^\alpha+n^\alpha}^{>0}>0,\end{aligned}$$

and whenever n is odd

$$\begin{aligned}\sum_{k=1}^n (-1)^k k^\alpha &= -1+\underbrace{2^\alpha-3^\alpha}_{<0}+\underbrace{4^\alpha-5^\alpha}_{<0}+\cdots-(n-2)^\alpha+\underbrace{(n-1)^\alpha-n^\alpha}_{<0}<0 \\ \sum_{k=1}^n (-1)^k k^\alpha &= \overbrace{-1+2^\alpha}^{>0}\;\overbrace{-3^\alpha+4^\alpha}^{>0}-5^\alpha+\cdots\overbrace{-(n-2)^\alpha+(n-1)^\alpha}^{>0}-n^\alpha>-n^\alpha.\end{aligned}$$

It follows that

$$\sum_{n=1}^\infty \frac{1}{n^p}\left|\sum_{k=1}^n (-1)^k k^\alpha\right|^p \leq \sum_{n=1}^\infty \frac{1}{n^p}\, n^{p\,\alpha} = \sum_{n=1}^\infty \frac{1}{n^{p\,(1-\alpha)}} < \infty$$

whenever $p\,(1-\alpha)>1$. Hence, for every $\alpha<1/q$, we have that $\left((-1)^n n^\alpha\right)_{n=1}^\infty \in [\mathcal{C},\ell^p]$ and thus $f_\alpha(s) := \sum_{n=1}^\infty (-1)^n n^{\alpha-s} \in \mathcal{H}([\mathcal{C},\ell^p])$. Since $\sigma_a(\mathcal{H}([\mathcal{C},\ell^p])\geq \sigma_a(f_\alpha)=1+\alpha$, making $\alpha\to 1/q$ it follows that $\sigma_a(\mathcal{H}([\mathcal{C},\ell^p])\geq 1+1/q$.

(b) Let $s_0 = \sigma + it \in \mathbb{C}_{1+1/q}$ and $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}([\mathcal{C}, \ell^p])$. By writing $a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k$, for all $n \geq 2$, we obtain

$$\begin{aligned}
|f(s_0)| &\leq \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \\
&\leq |a_1| + \sum_{n=2}^{\infty} n^{-\sigma} \left| \sum_{k=1}^{n} a_k \right| + \sum_{n=2}^{\infty} n^{-\sigma} \left| \sum_{k=1}^{n-1} a_k \right| \\
&\leq \sum_{n=1}^{\infty} n^{-\sigma} \left| \sum_{k=1}^{n} a_k \right| + \sum_{n=2}^{\infty} (n-1)^{-\sigma} \left| \sum_{k=1}^{n-1} a_k \right| \\
&= 2 \sum_{n=1}^{\infty} n^{-\sigma} \left| \sum_{k=1}^{n} a_k \right| \\
&= 2 \sum_{n=1}^{\infty} n^{1-\sigma} \frac{1}{n} \left| \sum_{k=1}^{n} a_k \right|.
\end{aligned}$$

So

$$|f(s_0)| \leq 2 \left(\sum_{n=1}^{\infty} \frac{1}{n^p} \left| \sum_{k=1}^{n} a_k \right|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} n^{(1-\sigma)q} \right)^{1/q},$$

that is,

$$|f(s_0)| \leq 2 \, \|f\|_{\mathcal{H}([\mathcal{C}, \ell^p])} \zeta(q(\sigma - 1))^{1/q}.$$

We conclude that

$$\|\delta_{s_0}\| \leq 2\zeta(q(\sigma - 1))^{1/q}.$$

□

Chapter 3

The multiplier algebra of $\mathcal{H}(ces_p)$

In this chapter we look into the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ of $\mathcal{H}(ces_p)$, that is, the space of all analytic functions f (on some domain containing $\mathbb{C}_{1/q}$) such that $fg \in \mathcal{H}(ces_p)$ for all $g \in \mathcal{H}(ces_p)$. The chapter is organized in four sections. In the first section we review general results for the multiplier algebra $\mathcal{M}(\mathcal{E})$ of a space $\mathcal{E}$ of Dirichlet series. The second section is devoted to the study of the multiplier algebra of the spaces $\mathcal{A}^r$ (for which $\mathcal{M}(\mathcal{A}^r) = \mathcal{A}^r$) and $\mathcal{A}^{2,r}$ (for which $\mathcal{M}(\mathcal{A}^{2,r}) = \mathcal{H}^\infty(\mathbb{C}_r)$). We also consider the cases of $\mathcal{M}(\mathcal{H}(\ell^p))$ and $\mathcal{M}(\mathcal{H}([\mathcal{C}, \ell^p]))$. The third section is the main section of the chapter, where we study $\mathcal{M}(\mathcal{H}(ces_p))$. We prove that monomials, and hence Dirichlet polynomials, are multipliers on $\mathcal{H}(ces_p)$ and calculate their norm as multipliers (3.4). This allows to deduce that $\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}(ces_p))$ (Theorem 3.11 and Proposition 3.3). We prove, via point evaluations, that multipliers on $\mathcal{H}(ces_p)$ are bounded functions on the vertical half-plane $\mathbb{C}_{1/q}$ (Theorem 3.11 and Proposition 3.2). This last result allows to show that the spaces $\mathcal{A}^{1/q}$, $\mathcal{M}(\mathcal{H}(ces_p))$ and $\mathcal{H}^\infty(\mathbb{C}_{1/q})$ have the same positive cone and that $\mathcal{A}^{1/q}$ is the solid core of $\mathcal{M}(\mathcal{H}(ces_p))$, with respect to the coefficient-wise order (Proposition 3.13). A fine use of abscissae of convergence allows to deduce that $\mathcal{M}(\mathcal{H}(ces_p)) \neq \mathcal{H}^\infty(\mathbb{C}_{1/q})$ (Theorem 3.11). At this stage we know that the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ satisfies

$$\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}(ces_p)) \subsetneq \mathcal{H}^\infty(\mathbb{C}_{1/q}),$$

and that $\mathcal{A}^{1/q}$ is the solid core of $\mathcal{M}(\mathcal{H}(ces_p))$. The fact that $\mathcal{M}(\mathcal{H}(ces_p)) \neq \mathcal{H}^\infty(\mathbb{C}_{1/q})$ shows a completely different situation to that of the multipliers algebras of the previously considered spaces of Dirichlet series, $\mathcal{H}$, $\mathcal{H}^p$, $\mathcal{H}_\alpha$, $\mathscr{A}^p_\mu$, $\mathscr{B}^p$, $\mathscr{D}_\alpha$. For these spaces the multiplier algebra is $\mathcal{H}^\infty$. This fact is in accordance and actually follows from the classical result of Schur identifying the multiplier algebra of the Hardy space $H^2(\mathbb{D})$ with the space $H^\infty(\mathbb{D})$, Theorem

1.24. However, it is noteworthy that if we consider the space of all Taylor series on the unit disc $\mathbb{D}$ with coefficient belonging to ces_p, it turns out that its multiplier algebra is not $H^\infty(\mathbb{D})$ but a rather smaller algebra, namely, the Wiener algebra of all absolutely convergent Taylor series; [24, Theorem 3.1], [25, Theorem 4.1]. This motivates the conjecture

$$\mathcal{M}(\mathcal{H}(ces_p)) = \mathcal{A}^{1/q},$$

that is, multipliers are Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ satisfying the condition $\sum_{n=1}^\infty |a_n| n^{-1/q} < \infty$. For establishing the conjecture we first prove it for multipliers $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ such that $a_n \neq 0$ only if the prime factorization of n has primes no other than the first r primes, $p_1, p_2, \ldots, p_r$, for a fixed $r \in \mathbb{N}$ (Theorem 3.15). The full identification is proven, with no restriction on the coefficients of the multipliers, in Theorem 3.17. In the last section we study the existence of compact multipliers (Theorem 3.20) and the sequence multipliers from $\mathcal{H}(ces_p)$ into its multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$, that is, we study how "close" is $\mathcal{H}(ces_p)$ from $\mathcal{M}(\mathcal{H}(ces_p))$ (Theorem 3.21).

The chapter is written in a somewhat inefficient way, at least from the viewpoint of conciseness. The reason for this choice is the following. The effort and time devoted to proving the identification of the multiplier algebra of $\mathcal{H}(ces_p)$ was largely augmented by the fact that most of the time we were led by a wrong conjecture and not pursuing the right result. It was, however, that effort and the detailed analysis that it required what eventually led to the correct result. This is presented in Theorem 3.17. The previous results which are presented correspond to the steps that helped formulating the right conjecture.

3.1 General facts on multiplier algebras

We collect several facts on the multipliers and multiplier algebra of a Banach space of Dirichlet series. Although some of them are known, we include the proofs for the sake of completeness.

Given a Banach space of Dirichlet series $\mathcal{E} \subseteq \mathcal{D}$ with convergence abscissa $\sigma_c(\mathcal{E})$, a *multiplier* on $\mathcal{E}$ is an analytic function f on $\mathbb{C}_{\sigma_c(\mathcal{E})}$ with the property that $fg \in \mathcal{E}$ for every $g \in \mathcal{E}$. The *multiplier algebra* of $\mathcal{E}$ is the space of all multipliers on $\mathcal{E}$, which will be denoted by $\mathcal{M}(\mathcal{E})$.

Proposition 3.1. *Let $\mathcal{E} \subseteq \mathcal{D}$ be a Banach space of Dirichlet series. Suppose that there exists $\sigma \geq \sigma_c(\mathcal{E})$ such that the point evaluation functional δ_{s_0} is continuous on $\mathcal{E}$ for every $s_0 \in \mathbb{C}_\sigma$. Then the following holds:*

(a) For every $f \in \mathcal{M}(\mathcal{E})$, the operator $M_f \colon \mathcal{E} \to \mathcal{E}$, given by

$$g \in \mathcal{E} \mapsto M_f(g) := fg \in \mathcal{E},$$

is linear and bounded.

(b) If the constant function $\mathbf{1} \in \mathcal{E}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ and for every $f \in \mathcal{M}(\mathcal{E})$ it follows that

$$\|f\|_{\mathcal{E}} \leq \|\mathbf{1}\|_{\mathcal{E}} \|f\|_{\mathcal{M}(\mathcal{E})}, \tag{3.1}$$

where $\|f\|_{\mathcal{M}(\mathcal{E})}$ denotes the operator norm of M_f. Moreover, in this case:

(b.1) $\mathcal{M}(\mathcal{E})$ is a closed subspace of the space $B(\mathcal{E})$ of all bounded linear operators of $\mathcal{E}$ into itself.

(b.2) Endowing $\mathcal{M}(\mathcal{E})$ with the norm $\|\cdot\|_{\mathcal{M}(\mathcal{E})}$, the containment $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ is continuous with embedding constant equal to $\|\mathbf{1}\|_{\mathcal{E}}$.

(b.3) For every $f \in \mathcal{M}(\mathcal{M}(\mathcal{E}))$, the operator $M_f \colon \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{E})$ defines a bounded linear operator and, with equality of norms, we have

$$\mathcal{M}(\mathcal{M}(\mathcal{E})) = \mathcal{M}(\mathcal{E}).$$

Proof. (a) We use the closed graph theorem. Let $f \in \mathcal{M}(\mathcal{E})$ and suppose that $g_n \to 0$ and $M_f(g_n) \to h$ in $\mathcal{E}$. For every $s_0 \in \mathbb{C}_\sigma$, since δ_{s_0} is continuous on $\mathcal{E}$, it follows that $g_n(s_0) \to 0$ and $f(s_0)g_n(s_0) = M_f(g_n)(s_0) \to h(s_0)$. Then $h = 0$ on $\mathbb{C}_\sigma$ and, by Theorem 1.13, $h = 0$. Consequently, the operator M_f is bounded.

(b) Suppose that $\mathbf{1} \in \mathcal{E}$. Then, for $f \in \mathcal{M}(\mathcal{E})$ we have that $f = M_f(\mathbf{1}) \in \mathcal{E}$ and

$$\|f\|_{\mathcal{E}} = \|M_f(\mathbf{1})\|_{\mathcal{E}} \leq \|\mathbf{1}\|_{\mathcal{E}} \|f\|_{\mathcal{M}(\mathcal{E})}.$$

(b.1) Let $(f_n)_{n=1}^\infty \subseteq \mathcal{M}(\mathcal{E})$ be such that $M_{f_n} \to T \in B(\mathcal{E})$ in the operator norm. Then, for each $g \in \mathcal{E}$ we have that $f_n \cdot g \to T(g)$ in $\mathcal{E}$ and so $f_n(s_0)g(s_0) \to T(g)(s_0)$ for all $s_0 \in \mathbb{C}_\sigma$. On the other hand, since $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{E}$ by (3.1), there exists $f \in \mathcal{E}$ such that $f_n \to f$ in $\mathcal{E}$ and so pointwise on $\mathbb{C}_\sigma$. Then, $T(g)(s_0) = f(s_0)g(s_0)$ for $s_0 \in \mathbb{C}_\sigma$ and thus $T(g) = f \cdot g$. Hence, $f \in \mathcal{M}(\mathcal{E})$ and $T = M_f$.

(b.2) is clear from the previous results.

(b.3) Since $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ continuously we have that the point evaluation δ_{s_0} is continuous on $\mathcal{M}(\mathcal{E})$ for every $s_0 \in \mathbb{C}_\sigma$. Noting that $\sigma \geq \sigma_c(\mathcal{E}) \geq \sigma_c(\mathcal{M}(\mathcal{E}))$, by (a), every $f \in \mathcal{M}(\mathcal{M}(\mathcal{E}))$ defines a bounded linear operator M_f on $\mathcal{M}(\mathcal{E})$. Since $\mathbf{1} \in \mathcal{M}(\mathcal{E})$, by (b.2) we have that $\mathcal{M}(\mathcal{M}(\mathcal{E})) \subseteq \mathcal{M}(\mathcal{E})$ and

$$\|f\|_{\mathcal{M}(\mathcal{E})} \leq \|f\|_{\mathcal{M}(\mathcal{M}(\mathcal{E}))}$$

for all $f \in \mathcal{M}(\mathcal{M}(\mathcal{E}))$, as $\|\mathbf{1}\|_{\mathcal{M}(\mathcal{E})} = 1$. Conversely, let $f \in \mathcal{M}(\mathcal{E})$. For every $g \in \mathcal{M}(\mathcal{E})$ and $h \in \mathcal{E}$, as $gh \in \mathcal{E}$, we have that $fgh \in \mathcal{E}$ and

$$\|fgh\|_{\mathcal{E}} \leq \|f\|_{\mathcal{M}(\mathcal{E})}\|gh\|_{\mathcal{E}} \leq \|f\|_{\mathcal{M}(\mathcal{E})}\|g\|_{\mathcal{M}(\mathcal{E})}\|h\|_{\mathcal{E}}.$$

Then, $fg \in \mathcal{M}(\mathcal{E})$ with $\|fg\|_{\mathcal{M}(\mathcal{E})} \leq \|f\|_{\mathcal{M}(\mathcal{E})}\|g\|_{\mathcal{M}(\mathcal{E})}$. Thus, $f \in \mathcal{M}(\mathcal{M}(\mathcal{E}))$ with $\|f\|_{\mathcal{M}(\mathcal{M}(\mathcal{E}))} \leq \|f\|_{\mathcal{M}(\mathcal{E})}$. □

Next proposition shows that, under minimal conditions which guarantee a good behavior of $\mathcal{M}(\mathcal{E})$, every multiplier on $\mathcal{E}$ is bounded on the aproppriate domain.

Proposition 3.2. *Let $\mathcal{E} \subseteq \mathcal{D}$ be a Banach space of Dirichlet series satisfying the condition of Proposition 3.1 for some $\sigma \geq \sigma_c(\mathcal{E})$ and such that $\mathbf{1} \in \mathcal{E}$. Then,*

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{H}^\infty(\mathbb{C}_\sigma),$$

where the containment is continuous with continuity constant equal to one.

Proof. Let $f \in \mathcal{M}(\mathcal{E})$. By Proposition 3.1, we have that $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ and so $f^2 = ff \in \mathcal{E}$ with

$$\|f^2\|_{\mathcal{E}} \leq \|f\|_{\mathcal{E}}\|f\|_{\mathcal{M}(\mathcal{E})} \leq \|\mathbf{1}\|_{\mathcal{E}}\|f\|^2_{\mathcal{M}(\mathcal{E})}.$$

Iterating the above procedure, we obtain, for every $n \geq 1$, that $f^n \in \mathcal{E}$ and

$$\|f^n\|_{\mathcal{E}} \leq \|\mathbf{1}\|_{\mathcal{E}}\|f\|^n_{\mathcal{M}(\mathcal{E})}.$$

For each $s_0 \in \mathbb{C}_\sigma$, by hypothesis, the point evaluation functional δ_{s_0} is bounded on $\mathcal{E}$. Then

$$\begin{aligned} |f^n(s_0)| &= |\delta_{s_0}(f^n)| \\ &\leq \|\delta_{s_0}\| \cdot \|f^n\|_{\mathcal{E}} \\ &\leq \|\delta_{s_0}\| \cdot \|\mathbf{1}\|_{\mathcal{E}}\|f\|^n_{\mathcal{M}(\mathcal{E})}. \end{aligned}$$

Since $|f^n(s_0)| = |f(s_0)|^n$, it follows that

$$|f(s_0)| \leq \Big(\|\delta_{s_0}\| \cdot \|\mathbf{1}\|_{\mathcal{E}}\Big)^{1/n}\|f\|_{\mathcal{M}(\mathcal{E})}.$$

Making $n \to \infty$ we have that $|f(s_0)| \leq \|f\|_{\mathcal{M}(\mathcal{E})}$. Hence, $f \in \mathcal{H}^\infty(\mathbb{C}_\sigma)$ and $\|f\|_{\mathcal{H}^\infty(\mathbb{C}_\sigma)} \leq \|f\|_{\mathcal{M}(\mathcal{E})}$. □

Moreover, if the monomials n^{-s}, for $n \geq 1$, are multipliers on $\mathcal{E}$, a weighted ℓ^1 space of Dirichlet series is contained in $\mathcal{M}(\mathcal{E})$.

Proposition 3.3. *Let $\mathcal{E} \subseteq \mathcal{D}$ be a Banach space of Dirichlet series satisfying the condition of Proposition 3.1 for some $\sigma \geq \sigma_c(\mathcal{E})$ and such that $\mathbf{1} \in \mathcal{E}$. If $\{n^{-s} : n \geq 1\} \subset \mathcal{M}(\mathcal{E})$, denoting $\mu_n = \|n^{-s}\|_{\mathcal{M}(\mathcal{E})}$, we have that*

$$\mathcal{A}((\mu_n)_{n=1}^{\infty}) := \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=1}^{\infty} |a_n|\mu_n < \infty \right\} \subseteq \mathcal{M}(\mathcal{E})$$

and $\|f\|_{\mathcal{M}(\mathcal{E})} \leq \sum_{n=1}^{\infty} |a_n|\mu_n$ for all $f \in \mathcal{A}((\mu_n)_{n=1}^{\infty})$.

Proof. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{A}((\mu_n)_{n=1}^{\infty})$. The series $\sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent in $\mathcal{M}(\mathcal{E})$, as

$$\sum_{n=1}^{\infty} \|a_n n^{-s}\|_{\mathcal{M}(\mathcal{E})} = \sum_{n=1}^{\infty} |a_n|\mu_n < \infty,$$

and so it converges in norm to some $h \in \mathcal{M}(\mathcal{E})$. Since, $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{E}$ continuously and so norm convergence in $\mathcal{M}(\mathcal{E})$ implies pointwise convergence on $\mathbb{C}_\sigma$, it follows that $f = h \in \mathcal{M}(\mathcal{E})$. From the equality above it follows that $\|f\|_{\mathcal{M}(\mathcal{E})} \leq \sum_{n=1}^{\infty} |a_n|\mu_n$. ☐

Under the natural conditions of Proposition 3.1, a multiplier on $\mathcal{E}$ is in fact a Dirichlet series in $\mathcal{E}$. The pointwise product $f(s)g(s)$ of two Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ is, in the appropriate vertical half-plane in which both series converge, the Dirichlet series $h(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ whose sequence of coefficients $c = (c_n)_{n=1}^{\infty}$ is given by the *Dirichlet convolution* $c := a \cdot b$ of the sequences $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$, that is,

$$c_n = (a \cdot b)_n := \sum_{k|n} a_k b_{\frac{n}{k}} = \sum_{k|n} a_{\frac{n}{k}} b_k, \quad n \geq 1,$$

where $k|n$ denotes that k is a divisor of n.

Remark 3.4. There is a matrix representation for the action on $\mathcal{E}$ of every Dirichlet series belonging to $\mathcal{M}(\mathcal{E})$. Given $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, consider the infinite matrix

$$A_f = (\alpha_{ij})_{i,j=1}^{\infty} = \begin{cases} \alpha_{ij} = a_{i/j} & \text{if } j|i, \\ 0 & \text{in other case,} \end{cases}$$

that is,

$$A_f = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_3 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_4 & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_5 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_6 & a_3 & a_2 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_7 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ a_8 & a_4 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & \cdots \\ a_9 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & \cdots \\ a_{10} & a_5 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & \cdots \\ a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & \cdots \\ a_{12} & a_6 & a_4 & a_3 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note, for every $b = (b_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}$, that

$$A_f b = \left(\sum_{k=1}^{\infty} \alpha_{nk} b_k \right)_{n=1}^{\infty} = \left(\sum_{k|n} a_{\frac{n}{k}} b_k \right)_{n=1}^{\infty}$$

and, for $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$,

$$M_f(g)(s) = f(s)g(s) = \sum_{n=1}^{\infty} \left(\sum_{k|n} a_{\frac{n}{k}} b_k \right) n^{-s}.$$

Considering the sequence space

$$\widehat{\mathcal{E}} := \left\{ (b_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} : \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{E} \right\},$$

it follows that $f \in \mathcal{M}(\mathcal{E})$, that is $M_f : \mathcal{E} \to \mathcal{E}$, if and only if $A_f : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$.

Remark 3.5. Consider the monomials $\{m^{-s} : m \geq 1\}$ associated to the canonical vectors $\{e^m : m \geq 1\}$. For a Dirichlet series $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, setting $b = (b_n)_{n=1}^{\infty}$, the coefficients of the Dirichlet series $m^{-s}g$ are given by the sequence $e^m \cdot b$. For each $n \in \mathbb{N}$ we have

$$\begin{aligned} (e^m \cdot b)_n = \sum_{k|n} e^m_k b_{\frac{n}{k}} &= \begin{cases} b_{\frac{n}{m}} & \text{if } m|n, \\ 0 & \text{in other case,} \end{cases} \\ &= \begin{cases} b_i & \text{if } n = im \text{ for some } i \geq 1, \\ 0 & \text{in other case,} \end{cases} \end{aligned}$$

that is,

$$e^m \cdot b = (0, ..., 0, \underbrace{b_1}_{m}, 0, ..., 0, \underbrace{b_2}_{2m}, 0, ..., 0, \underbrace{b_3}_{3m}, ...),$$

Hence, multiplying a Dirichlet series by the monomial m^{-s} "spreads out" by intervals of m steps the coefficients of the initial Dirichlet series.

3.2 First examples of multiplier algebras

The multiplier algebras of the spaces of Dirichlet series $\mathcal{A}^r$, $\mathcal{A}^{2,r}$ and $\mathcal{H}^\infty(\mathbb{C}_r)$ studied in Section 2.3 can be identified. The situation is much more complicated for the spaces $\mathcal{H}(\ell^p)$ and $\mathcal{H}([\mathcal{C}, \ell^p])$ studied in Section 2.4, nevertheless, although we have no identification for their algebras, we show some results related to them.

Let us start with the space $\mathcal{A}^r$, which turns out to coincide isometrically with its multiplier algebra.

Proposition 3.6. *The equality $\mathcal{M}(\mathcal{A}^r) = \mathcal{A}^r$ holds with equal norms.*

Proof. From Propositions 2.17 and 3.1, it follows that $\mathcal{M}(\mathcal{A}^r) \subseteq \mathcal{A}^r$ continuously with embedding constant equal to one.

For the converse containment consider the monomials $\{m^{-s} : m \geq 1\}$. Given $g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{A}^r$, setting $b = (b_n)_{n=1}^{\infty}$, we have that $e^m \cdot b$ is the sequence of coefficients of the Dirichlet series $m^{-s} g$ (see Remark 3.5). Then, since

$$\sum_{n=1}^{\infty} |(e^m \cdot b)_n| n^{-r} = \sum_{i=1}^{\infty} |(e^m \cdot b)_{im}| (im)^{-r} = m^{-r} \sum_{i=1}^{\infty} |b_i| i^{-r} = m^{-r} \|g\|_{\mathcal{A}^r},$$

it follows that $m^{-s} g \in \mathcal{A}^r$ and $\|m^{-s} g\|_{\mathcal{A}^r} = m^{-r} \|g\|_{\mathcal{A}^r}$. Hence, $m^{-s} \in \mathcal{M}(\mathcal{A}^r)$ and $\|m^{-s}\|_{\mathcal{M}(\mathcal{A}^r)} \leq m^{-r}$. Actually $\|m^{-s}\|_{\mathcal{M}(\mathcal{A}^r)} = m^{-r}$, as $\mathbf{1} \in \mathcal{A}^r$ with norm one and $\|m^{-s}\|_{\mathcal{A}^r} = m^{-r}$. Therefore, from Proposition 3.3, the containment $\mathcal{A}^r \subseteq \mathcal{M}(\mathcal{A}^r)$ holds continuously with embedding constant equal to one. □

The space $\mathcal{H}^\infty(\mathbb{C}_r)$ also coincides isometrically with its multiplier algebra, as can be directly checked, but moreover it coincides with the multiplier algebra of the space $\mathcal{A}^{2,r}$.

Proposition 3.7. *The equality $\mathcal{M}(\mathcal{A}^{2,r}) = \mathcal{H}^\infty(\mathbb{C}_r)$ holds with equal norms. Morever,*

$$\mathcal{A}^r \subsetneq \mathcal{M}(\mathcal{A}^{2,r}) = \mathcal{H}^\infty(\mathbb{C}_r) \subsetneq \mathcal{A}^{2,r},$$

where the containments are continuous with continuity constant equal to one.

Proof. From Propositions 2.18 and 3.1 it follows that $\mathcal{M}(\mathcal{A}^{2,r}) \subseteq \mathcal{A}^{2,r}$ continuously with embedding constant equal to one.

Consider the translation map $\tau_r : \mathcal{D} \to \mathcal{D}$ given by

$$\tau_r(f)(s) = f(s+r) = \sum_{n=1}^{\infty} (a_n n^{-r}) n^{-s}$$

for $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, (2.19). Note that $\tau_r(fg) = \tau_r(f)\tau_r(f)$ for all $f, g \in \mathcal{D}$. We have observed before that

$$\tau_r \colon \mathcal{H}^\infty(\mathbb{C}_r) \to \mathcal{H}^\infty(\mathbb{C}_0)$$

is an isometric isomorphism. Also, τ_r clearly induces an isometric isomorphism

$$\tau_r \colon \mathcal{A}^{2,r} \to \mathcal{A}^{2,0} = \mathcal{H}.$$

Let us show that $\mathcal{M}(\mathcal{A}^{2,r}) = \mathcal{H}^\infty(\mathbb{C}_r)$ by using translation map τ_r and the fact that $\mathcal{M}(\mathcal{H}) = \mathcal{H}^\infty(\mathbb{C}_0)$ with equality of norms (Theorem 1.4).

Let $f \in \mathcal{H}^\infty(\mathbb{C}_r)$ and $g \in \mathcal{A}^{2,r}$. Since $\tau_r(f) \in \mathcal{H}^\infty(\mathbb{C}_0) = \mathcal{M}(\mathcal{H})$ and $\tau_r(g) \in \mathcal{H}$, we have that $\tau_r(fg) = \tau_r(f)\tau_r(g) \in \mathcal{H}$ and so $fg \in \mathcal{A}^{2,r}$. Moreover,

$$\begin{aligned} \|fg\|_{\mathcal{A}^{2,r}} &= \|\tau_r(fg)\|_{\mathcal{H}} \\ &= \|\tau_r(f)\tau_r(g)\|_{\mathcal{H}} \\ &\le \|\tau_r(f)\|_{\mathcal{M}(\mathcal{H})} \|\tau_r(g)\|_{\mathcal{H}} \\ &= \|\tau_r(f)\|_{\mathcal{H}^\infty(\mathbb{C}_0)} \|g\|_{\mathcal{A}^{2,r}} \\ &= \|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)} \|g\|_{\mathcal{A}^{2,r}}. \end{aligned}$$

Then $f \in \mathcal{M}(\mathcal{A}^{2,r})$ with $\|f\|_{\mathcal{M}(\mathcal{A}^{2,r})} \le \|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)}$.

Let now $f \in \mathcal{M}(\mathcal{A}^{2,r})$ and $g \in \mathcal{H}$. Since $\tau_{-r}(g) \in \mathcal{A}^{2,r}$ we have that $f\tau_{-r}(g) \in \mathcal{A}^{2,r}$ and so $\tau_r(f)g = \tau_r(f\tau_{-r}(g)) \in \mathcal{H}$. Moreover,

$$\begin{aligned} \|\tau_r(f)g\|_{\mathcal{H}} &= \|\tau_r(f\tau_{-r}(g))\|_{\mathcal{H}} \\ &= \|f\tau_{-r}(g)\|_{\mathcal{A}^{2,r}} \\ &\le \|f\|_{\mathcal{M}(\mathcal{A}^{2,r})} \|\tau_{-r}(g)\|_{\mathcal{A}^{2,r}} \\ &= \|f\|_{\mathcal{M}(\mathcal{A}^{2,r})} \|g\|_{\mathcal{H}}. \end{aligned}$$

Then $\tau_r(f) \in \mathcal{M}(\mathcal{H}) = \mathcal{H}^\infty(\mathbb{C}_0)$, that is, $f \in \mathcal{H}^\infty(\mathbb{C}_r)$, and

$$\|f\|_{\mathcal{H}^\infty(\mathbb{C}_r)} = \|\tau_r(f)\|_{\mathcal{H}^\infty(\mathbb{C}_0)} = \|\tau_r(f)\|_{\mathcal{M}(\mathcal{H})} \le \|f\|_{\mathcal{M}(\mathcal{A}^{2,r})}.$$

The strictness of the inclusion $\mathcal{H}^\infty(\mathbb{C}_r) \subsetneq \mathcal{A}^{2,r}$ follows from Propositions 2.18 and 2.20 which state that $\sigma_c(\mathcal{A}^{2,r}) = \frac{1}{2} + r$ and $\sigma_c(\mathcal{H}^\infty(\mathbb{C}_r)) = r$.

We have seen in Section 2.3 that $\mathcal{A}^r \subseteq \mathcal{H}^\infty(\mathbb{C}_r)$ continuously with continuity constant equal to one, (2.20). Since, by Propositions 2.17 and 2.20, we have that $\sigma_a(\mathcal{A}^r) = r$ and $\sigma_a(\mathcal{H}^\infty(\mathbb{C}_r)) = r + \frac{1}{2}$, it follows that $\mathcal{A}^r \subsetneq \mathcal{H}^\infty(\mathbb{C}_r)$. $\square$

Consider now the space $\mathcal{H}(\ell^p)$. From Propositions 2.23 and 3.1 we have that $\mathcal{M}(\mathcal{H}(\ell^p)) \subseteq \mathcal{H}(\ell^p)$ continuously with embedding constant equal to one. The next proposition shows other inclusion relations.

Proposition 3.8. *For q being the conjugate exponent of p, the containments*

$$\mathcal{A}^0 \subseteq \mathcal{M}(\mathcal{H}(\ell^p)) \subsetneq \mathcal{H}^\infty(\mathbb{C}_{1/q})$$

hold continuously with continuity constant equal to one.

Proof. From Proposition 3.2 it follows that $\mathcal{M}(\mathcal{H}(\ell^p)) \subseteq \mathcal{H}^\infty(\mathbb{C}_{1/q})$ continuously with continuity constant equal to one. The inclusion is strict, as

$$\sigma_a(\mathcal{M}(\mathcal{H}(\ell^p))) \leq \sigma_a(\mathcal{H}(\ell^p)) = 1/q < 1/q + 1/2 = \sigma_a(\mathcal{H}^\infty(\mathbb{C}_{1/q})),$$

see Propositions 2.23 and 2.20.

For the first containment, let us consider the monomials $\{m^{-s} : m \geq 1\}$. Given $g(s) = \sum_{n=1}^\infty b_n n^{-s} \in \mathcal{H}(\ell^p)$, the coefficients of the Dirichlet series $m^{-s}g$ are given by the sequence $e^m \cdot b$, where $b = (b_n)_{n=1}^\infty$ (see Remark 3.5). Then, since

$$\sum_{n=1}^\infty |(e^m \cdot b)_n|^p = \sum_{i=1}^\infty |b_i|^p = \|g\|^p_{\mathcal{H}(\ell^p)},$$

it follows that $m^{-s}g \in \mathcal{H}(\ell^p)$ and $\|m^{-s}g\|_{\mathcal{H}(\ell^p)} = \|g\|_{\mathcal{H}(\ell^p)}$. Hence, $m^{-s} \in \mathcal{M}(\mathcal{H}(\ell^p))$ and $\|m^{-s}\|_{\mathcal{M}(\mathcal{H}(\ell^p))} \leq 1$. Actually $\|m^{-s}\|_{\mathcal{M}(\mathcal{H}(\ell^p))} = 1$, as $\mathbf{1} \in \mathcal{H}(\ell^p)$ with norm one and $\|m^{-s}\|_{\mathcal{H}(\ell^p)} = 1$. Therefore, from Proposition 3.3, the containment $\mathcal{A}^0 \subseteq \mathcal{M}(\mathcal{H}(\ell^p))$ holds continuously with embedding constant equal to one. $\square$

We finish this section by looking at the multiplier algebra of $\mathcal{H}([\mathcal{C}, \ell^p])$. Note that $\mathbf{1} \in \mathcal{H}([\mathcal{C}, \ell^p])$ with $\|\mathbf{1}\|_{\mathcal{H}([\mathcal{C},\ell^p])} = \zeta(p)^{1/p}$. From Propositions 2.24 and 3.1 we have that $\mathcal{M}(\mathcal{H}([\mathcal{C}, \ell^p])) \subseteq \mathcal{H}([\mathcal{C}, \ell^p])$ continuously with embedding constant $\zeta(p)^{1/p}$. Further inclusion relations follow in a similar way as for the multiplier algebra of $\mathcal{H}(\ell^p)$.

Proposition 3.9. *For q being the conjugate exponent of p, the containments*

$$\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}([\mathcal{C}, \ell^p])) \subsetneq \mathcal{H}^\infty(\mathbb{C}_{1+1/q})$$

hold continuously with continuity constants equal to one.

Proof. From Proposition 3.2 it follows that $\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p])) \subseteq \mathcal{H}^\infty(\mathbb{C}_{1+1/q})$ continuously with continuity constant equal to one. The inclusion is strict, as

$$\sigma_c(\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))) \leq \sigma_c(\mathcal{H}([\mathcal{C},\ell^p])) = 1 < 1 + 1/q = \sigma_c(\mathcal{H}^\infty(\mathbb{C}_{1+1/q})),$$

see Propositions 2.24 and 2.20.

Let us see that the monomials $\{m^{-s} : m \geq 1\}$ belong to $\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$. Let $g(s) = \sum_{n=1}^\infty b_n n^{-s} \in \mathcal{H}([\mathcal{C},\ell^p])$ and set $b = (b_n)_{n=1}^\infty$. The sequence $e^m \cdot b$ of coefficients of the Dirichlet series $m^{-s}g$ (see Remark 3.5) satisfies

$$\begin{aligned}
\sum_{n=1}^\infty \left|\frac{1}{n}\sum_{k=1}^n (e^m \cdot b)_k\right|^p &= \sum_{n=m}^\infty \left|\frac{1}{n}\sum_{i=1}^{\lfloor \frac{n}{m} \rfloor} b_i\right|^p \\
&= \sum_{j=1}^\infty \sum_{n=jm}^{(j+1)m-1} \left|\frac{1}{n}\sum_{i=1}^{j} b_i\right|^p \\
&= \sum_{j=1}^\infty \left|\sum_{i=1}^{j} b_i\right|^p \sum_{n=jm}^{(j+1)m-1} \frac{1}{n^p} \\
&\leq \sum_{j=1}^\infty \left|\sum_{i=1}^{j} b_i\right|^p \frac{m}{(jm)^p} \\
&= m^{1-p} \sum_{j=1}^\infty \left|\frac{1}{j}\sum_{i=1}^{j} b_i\right|^p \\
&= m^{1-p} \|g\|^p_{\mathcal{H}([\mathcal{C},\ell^p])},
\end{aligned}$$

and so $m^{-s}g \in \mathcal{H}([\mathcal{C},\ell^p])$ with $\|m^{-s}g\|_{\mathcal{H}([\mathcal{C},\ell^p])} \leq m^{-1/q}\|g\|_{\mathcal{H}([\mathcal{C},\ell^p])}$. Then, $m^{-s} \in \mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$ and $\|m^{-s}\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} \leq m^{-1/q}$. Actually,

$$\|m^{-s}\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} = m^{-1/q}.$$

Indeed, for every $j \geq 2$, since

$$\|j^{-s}\|^p_{\mathcal{H}([\mathcal{C},\ell^p])} = \sum_{n=j}^\infty \frac{1}{n^p} \leq \frac{1}{(p-1)(j-1)^{p-1}}$$

and

$$\|m^{-s}j^{-s}\|^p_{\mathcal{H}([\mathcal{C},\ell^p])} = \|(mj)^{-s}\|^p_{\mathcal{H}([\mathcal{C},\ell^p])} = \sum_{n=jm}^\infty \frac{1}{n^p} \geq \frac{1}{(p-1)(jm)^{p-1}},$$

we have that

$$\|m^{-s}\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} \geq \frac{\|m^{-s}j^{-s}\|_{\mathcal{H}([\mathcal{C},\ell^p])}}{\|j^{-s}\|_{\mathcal{H}([\mathcal{C},\ell^p])}} \geq \left(\frac{j-1}{jm}\right)^{1/q}.$$

Making $j \to \infty$, we arrive at $\|m^{-s}\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} \geq m^{-1/q}$.

Therefore, from Proposition 3.3, the containment $\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$ holds continuously with embedding constant equal to one. $\square$

3.3 The multiplier algebra of $\mathcal{H}(ces_p)$

The space of multipliers on $\mathcal{H}(ces_p)$ is the central part of this memoir, so we will pay special attention to the details. Recall that $1 < p < \infty$ and $1/p + 1/q = 1$.

We have seen that $\sigma_c(\mathcal{H}(ces_p)) = 1/q$ and that, for each $s_0 \in \mathbb{C}_{1/q}$, the point evaluation functional δ_{s_0} is continuous on $\mathcal{H}(ces_p)$, see Proposition 2.12 and Theorem 2.13. With these conditions, Proposition 3.1 implies that every multiplier f on $\mathcal{H}(ces_p)$ defines a bounded multiplication operator M_f from $\mathcal{H}(ces_p)$ into itself:

$$g \in \mathcal{H}(ces_p) \mapsto M_f(g) = fg \in \mathcal{H}(ces_p).$$

Moreover, since the constant function $\mathbf{1} \in \mathcal{H}(ces_p)$ and $\|\mathbf{1}\|_{\mathcal{H}(ces_p)} = \zeta(p)^{1/p}$, it also follows that

$$\mathcal{M}(\mathcal{H}(ces_p)) \subseteq \mathcal{H}(ces_p) \tag{3.2}$$

continuously with embedding constant $\zeta(p)^{1/p}$, that is,

$$\|f\|_{\mathcal{H}(ces_p)} \leq \zeta(p)^{1/p}\|f\|_{\mathcal{M}(\mathcal{H}(ces_p))}, \quad f \in \mathcal{M}(\mathcal{H}(ces_p)). \tag{3.3}$$

So, a multiplier f on $\mathcal{H}(ces_p)$ is actually a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ belonging to $\mathcal{H}(ces_p)$ and the action of the multiplication operator M_f on $g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{H}(ces_p)$ is given by

$$M_f(s) = f(s)g(s) = \sum_{n=1}^{\infty}\left(\sum_{k|n} a_k b_{\frac{n}{k}}\right)n^{-s}.$$

The boundedness of the operator M_f corresponds to the existence of some constant $M > 0$ such that

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\left|\sum_{j|k} a_j b_{\frac{k}{j}}\right|\right)^p\right)^{1/p} \leq M \cdot \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|b_k|\right)^p\right)^{1/p}$$

for all $g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{H}(ces_p)$. The least of such constants M is the operator norm $\|M_f\|$ of M_f as a bounded operator from $\mathcal{H}(ces_p)$ into itself, which we denote by $\|f\|_{\mathcal{M}(\mathcal{H}(ces_p))}$.

We first check that Dirichlet polynomials $\sum_{n=1}^{N} a_n n^{-s}$ are multipliers on $\mathcal{H}(ces_p)$. For this, in view of the linearity of the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$, it suffices to verify that the monomials $\{m^{-s} : m \geq 1\}$ belong to $\mathcal{M}(\mathcal{H}(ces_p))$. We will use the following result related to multipliers on $\mathcal{H}([\mathcal{C}, \ell^p])$ with positive coefficients.

Proposition 3.10. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a multiplier on $\mathcal{H}([\mathcal{C}, \ell^p])$ such that $a_n \geq 0$ for all $n \geq 1$. Then f is a multiplier on $\mathcal{H}(ces_p)$ and*

$$\|f\|_{\mathcal{M}(\mathcal{H}(ces_p))} \leq \|f\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))}.$$

Proof. For $g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{H}(ces_p)$, take $h(s) = \sum_{n=1}^{\infty} |b_n| n^{-s}$ which belongs to $\mathcal{H}([\mathcal{C}, \ell^p])$ with $\|h\|_{\mathcal{H}([\mathcal{C},\ell^p])} = \|g\|_{\mathcal{H}(ces_p)}$. Then $fh \in \mathcal{H}([\mathcal{C}, \ell^p])$. Since $a_n \geq 0$ for all $n \geq 1$, we have that

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \left| \sum_{j|k} a_j b_{\frac{k}{j}} \right| \right)^p \leq \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \left(\sum_{j|k} a_j |b_{\frac{k}{j}}| \right) \right)^p = \|fh\|_{\mathcal{H}([\mathcal{C},\ell^p])}^p$$

and so $fg \in \mathcal{H}(ces_p)$ with

$$\begin{aligned} \|fg\|_{\mathcal{H}(ces_p)} &\leq \|fh\|_{\mathcal{H}([\mathcal{C},\ell^p])} \\ &\leq \|f\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} \|h\|_{\mathcal{H}([\mathcal{C},\ell^p])} \\ &= \|f\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} \|g\|_{\mathcal{H}(ces_p)}. \end{aligned}$$

Hence, the conclusion follows. $\square$

We have seen in Proposition 3.9 that, for every $m \geq 1$, the monomial $m^{-s} \in \mathcal{M}(\mathcal{H}([\mathcal{C}, \ell^p]))$ with $\|m^{-s}\|_{\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))} = m^{-1/q}$. Then, the previous proposition gives that $m^{-s} \in \mathcal{M}(\mathcal{H}(ces_p))$ with $\|m^{-s}\|_{\mathcal{M}(\mathcal{H}(ces_p))} \leq m^{-1/q}$. Actually,

$$\|m^{-s}\|_{\mathcal{M}(\mathcal{H}(ces_p))} = m^{-1/q}, \tag{3.4}$$

as

$$\|m^{-s}\|_{\mathcal{M}(\mathcal{H}(ces_p))} \geq \frac{\|m^{-s} j^{-s}\|_{\mathcal{H}(ces_p)}}{\|j^{-s}\|_{\mathcal{H}(ces_p)}} = \frac{\|m^{-s} j^{-s}\|_{\mathcal{H}([\mathcal{C},\ell^p])}}{\|j^{-s}\|_{\mathcal{H}([\mathcal{C},\ell^p])}} \geq \left(\frac{j-1}{jm} \right)^{1/q}$$

for all $j \geq 2$, see the end of the proof of Proposition 3.9 for the last inequality.

Not only the Dirichlet polynomials $\sum_{n=1}^{N} a_n n^{-s}$ are included in $\mathcal{M}(\mathcal{H}(ces_p))$ but also the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ which are absolutely convergent for the norm of $\mathcal{M}(\mathcal{H}(ces_p))$. On the other hand multipliers on $\mathcal{H}(ces_p)$ are necessarily bounded functions in an adequate vertical half-plane.

Theorem 3.11. *The inclusions*

$$\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}(ces_p)) \subsetneq \mathcal{H}^\infty(\mathbb{C}_{1/q})$$

holds continuously with embedding constants equal to one.

Proof. The first containment follows from (3.4) and Proposition 3.3. The second one comes from Proposition 3.2.

It only remains to prove that $\mathcal{M}(\mathcal{H}(ces_p)) \neq \mathcal{H}^\infty(\mathbb{C}_{1/q})$. For this we calculate the abscissae of convergence and absolute convergence of $\mathcal{M}(\mathcal{H}(ces_p))$. From Proposition 2.17, Theorem 2.12 and the inclusions

$$\mathcal{A}^{1/q} \subseteq \mathcal{M}(\mathcal{H}(ces_p)) \subseteq \mathcal{H}(ces_p),$$

it follows that

$$\begin{aligned}\frac{1}{q} = \sigma_c(\mathcal{A}^{1/q}) &\leq \sigma_c\big(\mathcal{M}(\mathcal{H}(ces_p))\big)\\ &\leq \sigma_a\big(\mathcal{M}(\mathcal{H}(ces_p))\big) \leq \sigma_a(\mathcal{H}(ces_p)) = \frac{1}{q}.\end{aligned}$$

Hence, $\sigma_c\big(\mathcal{M}(\mathcal{H}(ces_p))\big) = \sigma_a\big(\mathcal{M}(\mathcal{H}(ces_p))\big) = 1/q$. Since, by Proposition 2.20, we have that $\sigma_a(\mathcal{H}^\infty(\mathbb{C}_{1/q})) = 1/q + 1/2$, it must be $\mathcal{M}(\mathcal{H}(ces_p)) \neq \mathcal{H}^\infty(\mathbb{C}_{1/q})$. □

Remark 3.12. The first containment in Theorem 3.11 allows to prove that

$$\mathcal{M}(\mathcal{H}(ces_{p_1})) \subseteq \mathcal{M}(\mathcal{H}(ces_{p_2})).$$

whenever $1 < p_1 < p_2 < \infty$. Let $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{M}(\mathcal{H}(ces_{p_1}))$ and take $\varepsilon > 0$ such that $1/q_1 + \varepsilon < 1/q_2$. Then, since $\sigma_c(\mathcal{M}(\mathcal{H}(ces_{p_1}))) = 1/q_1$, we have that

$$\sum_{n=1}^\infty |a_n| n^{-1/q_2} \leq \sum_{n=1}^\infty |a_n| n^{-(1/q_1)-\varepsilon} < \infty.$$

Thus, $f \in \mathcal{A}^{1/q_2} \subseteq M(\mathcal{H}(ces_{p_2}))$.

From Proposition 2.21 the positive cones of $\mathcal{A}^{1/q}$ and $\mathcal{H}^\infty(\mathbb{C}_{1/q})$ coincide. This fact together with Theorem 3.11 allows to identify the solid core of the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$.

Proposition 3.13. *Let $f \in \mathcal{M}(\mathcal{H}(ces_p))$ with $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ such that $a_n \geq 0$, for $n \geq 1$. Then $f \in \mathcal{A}^{1/q}$ and*

$$\|f\|_{\mathcal{A}^{1/q}} = \|f\|_{\mathcal{M}(\mathcal{H}(ces_p))} = \|f\|_{\mathcal{H}^\infty(\mathbb{C}_{1/q})}.$$

Thus, the positive cones of $\mathcal{A}^{1/q}$, $\mathcal{M}(\mathcal{H}(ces_p))$ and $\mathcal{H}^\infty(\mathbb{C}_{1/q})$ coincide. Moreover, the solid core of $\mathcal{M}(\mathcal{H}(ces_p))$ is the space $\mathcal{A}^{1/q}$.

Proof. If $f \in \mathcal{M}(\mathcal{H}(ces_p))$ is a Dirichlet series with positive coefficients, since $\mathcal{M}(\mathcal{H}(ces_p)) \subseteq \mathcal{H}^\infty(\mathbb{C}_{1/q})$, from Proposition 2.21 we have that $f \in \mathcal{A}^{1/q}$ and $\|f\|_{\mathcal{A}^{1/q}} = \|f\|_{\mathcal{H}^\infty(\mathbb{C}_{1/q})}$. Then the three norms coincide by Theorem 3.11.

Every solid subspace $\mathcal{E}$ of $\mathcal{M}(\mathcal{H}(ces_p))$ is a solid subspace of $\mathcal{H}^\infty(\mathbb{C}_{1/q})$ and so, from Proposition 2.21, $\mathcal{E} \subseteq \mathcal{A}^{1/q}$. □

Remark 3.14. (i) The multiplier space $\mathcal{M}(\mathcal{H}(ces_p))$ is a solid space of Dirichlet series if and only if it coincides with the space $\mathcal{A}^{1/q}$.

(ii) Since $\mathcal{M}(\mathcal{H}(ces_p))$ and $\mathcal{H}^\infty(\mathbb{C}_{1/q})$ have the same positive cone, in order to exhibit an example of Dirichlet series in $\mathcal{M}(\mathcal{H}(ces_p))$ but not in $\mathcal{H}^\infty(\mathbb{C}_{1/q})$, it must be taken into account that it cannot have all their coefficients positive.

Theorem 3.11 shows that the situation concerning the multiplier algebra of $\mathcal{H}(ces_p)$ is different from that of other spaces of Dirichlet series studied previously in the literature: in this case the multiplier algebra will not coincide with an algebra of bounded Dirichlet series.

A natural conjecture, motivated by Theorem 3.11, is that

$$\mathcal{M}(\mathcal{H}(ces_p)) = \mathcal{A}^{1/q}, \tag{3.5}$$

with equality of norms. As explained in the Introduction, this conjecture is to some extent also motivated by the case of the space $H(\mathbb{D}, ces_p)$, of Taylor series on the unit disc $\mathbb{D}$ of the complex plane with coefficients belonging to ces_p. It was proven by Curbera and Ricker that the multiplier space $\mathcal{M}(H(\mathbb{D}, ces_p))$ is the Wiener algebra of absolutely convergent Taylor series, which is the smallest algebra inside $H(\mathbb{D}, ces_p)$ which contains the polynomials, [24, Theorem 3.1], [25, Theorem 4.1].

The next result is a partial step in the direction of establishing conjecture (3.5).

Recall from Section 2.2, for $r \in \mathbb{N}$, the subset of $\mathbb{N}$ given by

$$\mathbb{N}_r := \Big\{ n = \prod_{i=1}^{r} p_i^{\alpha_i} : \ \alpha_1, \dots, \alpha_r \geq 0 \Big\},$$

where $p_1, ..., p_r$ denote the first r prime numbers.

Theorem 3.15. *Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a multiplier on $\mathcal{H}(ces_p)$. Suppose that*

$$a_n = 0, \quad n \notin \mathbb{N}_r.$$

Then $f \in \mathcal{A}^{1/q}$ and, moreover,

$$\|f\|_{\mathcal{A}^{1/q}} = \|f\|_{\mathcal{M}(\mathcal{H}(ces_p))}.$$

For the proof of Theorem 3.15 we need the next lemma, which we proof afterwards. For $r \in \mathbb{N}$, consider the set

$$I_r := \big\{k \in \mathbb{N} : \, p_i \nmid k \text{ for all } 1 \leq i \leq r\big\},$$

where $p_i \nmid k$ means that p_i is not a divisor of k. Note that $I_r \cap \mathbb{N}_r = \{1\}$. Define

$$\begin{aligned} A_r &:= \prod_{i=1}^{r} \Big(1 - \frac{1}{p_i}\Big), \\ B_r &:= 2^{r-1}. \end{aligned}$$

Lemma 3.16. *Let $0 < \alpha < 1$. For all $r, n \in \mathbb{N}$ we have*

$$A_r(n+1)^\alpha - B_r \leq \alpha \sum_{\substack{k=1 \\ k \in I_r}}^{n} \frac{1}{k^{1-\alpha}} \leq A_r n^\alpha + B_r. \tag{3.6}$$

Note that $A_r(n+1)^\alpha - B_r \geq 0$ precisely when $n+1 \geq (B_r/A_r)^{1/\alpha}$.

Proof of Theorem 3.15. Choose any α satisfying $1/(2q) < \alpha < 1/q$. Consider the sequence $b^\alpha = (b_n^\alpha)_{n=1}^\infty$ defined by

$$b_n^\alpha := \begin{cases} 1/n^{(1-\alpha)} & \text{if } n \in I_r \\ 0 & \text{in other case,} \end{cases} \tag{3.7}$$

and let $g^\alpha(s) := \sum_{n=1}^\infty b_n^\alpha n^{-s}$.

By Lemma 3.16 we have

$$\begin{aligned} \|g^\alpha\|^p_{\mathcal{H}(ces_p)} = \|b^\alpha\|^p_{ces_p} &= \sum_{n=1}^\infty \frac{1}{n^p} \Big(\sum_{k=1}^{n} |b_k^\alpha|\Big)^p \\ &= \sum_{n=1}^\infty \frac{1}{n^p} \Big(\sum_{\substack{k=1 \\ k \in I_r}}^{n} \frac{1}{k^{1-\alpha}}\Big)^p \\ &\leq \frac{1}{\alpha^p} \sum_{n=1}^\infty \frac{1}{n^p} \Big(A_r n^\alpha + B_r\Big)^p \\ &\leq \frac{A_r^p}{\alpha^p} \sum_{n=1}^\infty \frac{1}{n^{p(1-\alpha)}} \Big(1 + \frac{B_r}{A_r n^\alpha}\Big)^p \\ &\leq \frac{A_r^p}{\alpha^p} \sum_{n=1}^\infty \frac{1}{n^{p(1-\alpha)}} \Big(1 + \frac{K}{n^\alpha}\Big), \end{aligned}$$

where K is a constant, depending solely on p and r, obtained via the mean value theorem applied to the function $(1+x)^p$:

$$\begin{aligned}\Big(1+\frac{B_r}{A_r n^\alpha}\Big)^p &\le 1+\frac{B_r}{A_r n^\alpha}\max\Big\{p\big(1+x\big)^{p-1} : 0<x<\frac{B_r}{A_r n^\alpha}\Big\}\\ &\le 1+\frac{1}{n^\alpha}\Big(\frac{B_r}{A_r}p\Big(1+\frac{B_r}{A_r}\Big)^{p-1}\Big)\\ &= 1+\frac{K}{n^\alpha}.\end{aligned}$$

Thus,

$$\|g^\alpha\|^p_{\mathcal{H}(ces_p)} \le \frac{A_r^p}{\alpha^p}\Big(\zeta(p(1-\alpha))+K\zeta(p(1-\alpha)+\alpha)\Big). \tag{3.8}$$

On other hand, observe that each $k\in\mathbb{N}$ can be written as $k=\omega\,\gamma$ with $\omega\in\mathbb{N}_r$ and $\gamma\in I_r$. Then

$$(a\cdot b^\alpha)_k=\sum_{j|k}a_j b^\alpha_{\frac{k}{j}}=\sum_{\substack{j|k\\ j\in\mathbb{N}_r}}a_j b^\alpha_{\frac{k}{j}}=a_\omega b^\alpha_\gamma. \tag{3.9}$$

Indeed, if $j\in\mathbb{N}_r$ is such that $j|k$, it must then be that $j|\omega$. Moreover, if $j\neq\omega$ we have that $k/j\notin I_r$ and so $b^\alpha_{k/j}=0$.

Fix $m\in\mathbb{N}$ and consider the set

$$\mathbb{N}_r^m := \Big\{n=\prod_{i=1}^r p_i^{\alpha_i} \,:\, 0\le\alpha_i\le m \text{ for all } 1\le i\le r\Big\}.$$

Since $\omega\gamma=\hat\omega\hat\gamma$, with $\omega,\hat\omega\in\mathbb{N}_r$ and $\gamma,\hat\gamma\in I_r$, implies that $\omega=\hat\omega$ and $\gamma=\hat\gamma$, it follows that the set

$$\mathcal{O}=\bigcup_{\omega\in\mathbb{N}_r^m}\omega\cdot I_r$$

is a finite union of disjoint sets. Then,

$$\begin{aligned}
\|fg^\alpha\|^p_{\mathcal{H}(ces_p)} = \|a\cdot b^\alpha\|^p_{ces_p} &= \sum_{n=1}^{\infty}\frac{1}{n^p}\Big(\sum_{k=1}^{n}|(a\cdot b^\alpha)_k|\Big)^p \\
&\geq \sum_{n=p_r^{rm}}^{\infty}\frac{1}{n^p}\Big(\sum_{\substack{k=1\\ k\in\mathcal{O}}}^{n}|(a\cdot b^\alpha)_k|\Big)^p \\
&= \sum_{n=p_r^{rm}}^{\infty}\frac{1}{n^p}\Big(\sum_{\omega\in\mathbb{N}_r^m}\sum_{\substack{k=1\\ k\in\omega\cdot I_r}}^{n}|(a\cdot b^\alpha)_k|\Big)^p \\
&= \sum_{n=p_r^{rm}}^{\infty}\frac{1}{n^p}\Big(\sum_{\omega\in\mathbb{N}_r^m}\sum_{\substack{\gamma=1\\ \gamma\in I_r}}^{\lfloor n/\omega\rfloor}|(a\cdot b^\alpha)_{\omega\gamma}|\Big)^p \\
&= \sum_{n=p_r^{rm}}^{\infty}\frac{1}{n^p}\Big(\sum_{\omega\in\mathbb{N}_r^m}|a_\omega|\sum_{\substack{\gamma=1\\ \gamma\in I_r}}^{\lfloor n/\omega\rfloor}|b_\gamma^\alpha|\Big)^p \\
&= \sum_{n=p_r^{rm}}^{\infty}\frac{1}{n^p}\Big(\sum_{\omega\in\mathbb{N}_r^m}|a_\omega|\sum_{\substack{\gamma=1\\ \gamma\in I_r}}^{\lfloor n/\omega\rfloor}\frac{1}{\gamma^{1-\alpha}}\Big)^p.
\end{aligned}$$

Note that $\omega\leq p_r^{rm}$ whenever $\omega\in\mathcal{P}_r^m$, and so $n/\omega\geq 1$ for $n\geq p_r^{rm}$.

In order to apply Lemma 3.16 we restrict to $n\geq n_0 := 2^{4qr+2q(m-1)}p_r^{rm}$. In this case, for every $\omega\in\mathbb{N}_r^m$, we have

$$\begin{aligned}
\frac{B_r\omega^\alpha}{A_r} &= \frac{2^{r-1}\omega^\alpha}{\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)} \\
&\leq \frac{2^{r-1}\omega^\alpha}{\prod_{i=1}^{r}\frac{1}{2}} \\
&= 2^{2r-1}\omega^\alpha \\
&\leq \frac{2^{2r+(m-1)}}{2^m}p_r^{\alpha rm} \\
&\leq \frac{2^{4qr\alpha+2q\alpha(m-1)}}{2^m}p_r^{\alpha rm} \\
&\leq \frac{n^\alpha}{2^m},
\end{aligned}$$

since $1<2q\alpha$, as we are assuming that $1/(2q)<\alpha<1/q$. Consequently

$$1-\frac{B_r\omega^\alpha}{A_r n^\alpha}\geq 1-\frac{1}{2^m}$$

and, from Lemma 3.16, we have

$$\begin{aligned}\sum_{\substack{\gamma=1\\ \gamma\in I_r}}^{\lfloor n/\omega\rfloor}\frac{1}{\gamma^{1-\alpha}} &\geq \frac{A_r(\lfloor\frac{n}{\omega}\rfloor+1)^\alpha}{\alpha}-\frac{B_r}{\alpha}\\ &\geq \frac{A_r n^\alpha}{\alpha\omega^\alpha}\Big(1-\frac{B_r\omega^\alpha}{A_r n^\alpha}\Big)\\ &\geq \frac{A_r n^\alpha}{\alpha\omega^\alpha}\Big(1-\frac{1}{2^m}\Big).\end{aligned}$$

Then

$$\begin{aligned}\|fg^\alpha\|^p_{\mathcal{H}(ces_p)} &\geq \sum_{n=n_0}^{\infty}\frac{1}{n^p}\Big(\sum_{\omega\in\mathbb{N}_r^m}|a_\omega|\frac{A_r n^\alpha}{\alpha\omega^\alpha}\Big(1-\frac{1}{2^m}\Big)\Big)^p\\ &\geq \Big(1-\frac{1}{2^m}\Big)^p\frac{A_r^p}{\alpha^p}\Big(\sum_{n=n_0}^{\infty}\frac{1}{n^{p(1-\alpha)}}\Big)\Big(\sum_{\omega\in\mathbb{N}_r^m}\frac{|a_\omega|}{\omega^\alpha}\Big)^p, \qquad (3.10)\\ &\geq \Big(1-\frac{1}{2^m}\Big)^p\frac{A_r^p}{\alpha^p(p(1-\alpha)-1)n_0^{p(1-\alpha)-1}}\Big(\sum_{\omega\in\mathbb{N}_r^m}\frac{|a_\omega|}{\omega^\alpha}\Big)^p.\end{aligned}$$

From (3.8) and (3.10) we have

$$\begin{aligned}\|f\|_{\mathcal{M}(\mathcal{H}(ces_p))} \geq& \frac{\|fg^\alpha\|_{\mathcal{H}(ces_p)}}{\|g^\alpha\|_{\mathcal{H}(ces_p)}}\\ =&\Big(1-\frac{1}{2^m}\Big)\Big(\sum_{\omega\in\mathbb{N}_r^m}\frac{|a_\omega|}{\omega^\alpha}\Big)M(\alpha)^{1/p}, \qquad (3.11)\end{aligned}$$

for

$$\begin{aligned}M(\alpha)^{-1} :=&(p(1-\alpha)-1)(2^{4qr+2q(m-1)}p_r^{rm})^{p(1-\alpha)-1}\\ &\cdot\Big(\zeta(p(1-\alpha))+K\zeta(p(1-\alpha)+\alpha)\Big).\end{aligned}$$

Inequality (3.11) holds for every α with $1/(2q)<\alpha<1/q$. Thus, we can take limit when $\alpha\to(1/q)^-$. Since $p(1-\alpha)\to p(1-1/q)=1$, it follows that

$$\lim_{\alpha\to(1/q)^-}M(\alpha)=\lim_{\alpha\to(1/q)^-}(p(1-\alpha)-1)\zeta(p(1-\alpha))=1.$$

Consequently,

$$\begin{aligned}\|f\|_{\mathcal{M}(\mathcal{H}(ces_p))} \geq& \lim_{\alpha\to(1/q)^-}\Big(1-\frac{1}{2^m}\Big)\Big(\sum_{\omega\in\mathbb{N}_r^m}\frac{|a_\omega|}{\omega^\alpha}\Big)M(\alpha)^{1/p}\\ =&\Big(1-\frac{1}{2^m}\Big)\Big(\sum_{\omega\in\mathbb{N}_r^m}\frac{|a_\omega|}{\omega^{1/q}}\Big).\end{aligned}$$

Since the above inequality holds for arbitrary $m \in \mathbb{N}$, it follows that

$$\|f\|_{\mathcal{M}(\mathcal{H}(ces_p))} \geq \sum_{\omega\in\mathbb{N}_r} \frac{|a_\omega|}{\omega^{1/q}}.$$

$\square$

Proof of Lemma 3.16. We fix all throughout the proof $0 < \alpha < 1$.

We first establish (3.6) in the case when $r, n \in \mathbb{N}$ satisfy $n \leq p_r$. In this case, $I_r \cap \{1, 2, .., p_r\} = \{1\}$, so we have

$$\sum_{\substack{k=1\\ k\in I_r}}^{n} \frac{1}{k^{1-\alpha}} = 1.$$

The inequalities $\alpha \leq B_r \leq A_r n^\alpha + B_r$ establish the right-hand side of (3.6). For the left-hand side, direct calculation shows, when $r = 1$, that

$$A_r(p_r + 1)^\alpha - B_r \leq \alpha.$$

For $r \geq 2$ we have

$$\begin{aligned} A_r(n+1)^\alpha - B_r &\leq \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)(p_r + 1)^\alpha - 2^{r-1} \\ &\leq \frac{p_r + 1}{3} - 2^{r-1} \\ &\leq \frac{2^r + 1}{3} - 2^{r-1} \\ &= \frac{1}{3}(1 - 2^{r-1}) < 0, \end{aligned}$$

where we use that $p_r \leq 2^r$. Hence, (3.6) holds when $n \leq p_r$.

Next we consider the case when $n \geq p_r + 1$. We will prove it by induction on r. For $r = 1$ and $n \geq p_1 + 1 = 3$ we have to show that

$$\frac{(n+1)^\alpha}{2} - 1 \leq \alpha \sum_{\substack{k=1\\ k\in I_1}}^{n} \frac{1}{k^{1-\alpha}} \leq \frac{n^\alpha}{2} + 1. \tag{3.12}$$

Observe that $I_1 = \big\{k \in \mathbb{N} : 2 \nmid k\big\} = \big\{k \in \mathbb{N} : k \text{ is odd}\big\}$. Then

$$\begin{aligned} \sum_{\substack{k=1\\ k\in I_1}}^{n} \frac{1}{k^{1-\alpha}} &= \sum_{k=1}^{n} \frac{1}{k^{1-\alpha}} - \sum_{\substack{k=1\\ k \text{ even}}}^{n} \frac{1}{k^{1-\alpha}} \\ &= \sum_{k=1}^{n} \frac{1}{k^{1-\alpha}} - \frac{1}{2^{1-\alpha}} \sum_{j=1}^{\lfloor n/2\rfloor} \frac{1}{j^{1-\alpha}}. \end{aligned}$$

Since

$$(n+1)^\alpha - 1 \le \alpha \sum_{k=1}^{n} 1/k^{1-\alpha} \le n^\alpha, \quad n \in \mathbb{N},$$

it follows that

$$\begin{aligned}
\alpha \sum_{\substack{k=1\\ k\in I_1}}^{n} \frac{1}{k^{1-\alpha}} &\le n^\alpha - \frac{1}{2^{1-\alpha}}\left(\left(\left\lfloor \frac{n}{2}\right\rfloor + 1\right)^\alpha - 1\right)\\
&\le n^\alpha - \frac{1}{2^{1-\alpha}}\left(\frac{n^\alpha}{2^\alpha} - 1\right)\\
&= \frac{n^\alpha}{2} + \frac{1}{2^{1-\alpha}}\\
&\le \frac{n^\alpha}{2} + 1.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
\alpha \sum_{\substack{k=1\\ k\in I_1}}^{n} \frac{1}{k^{1-\alpha}} &\ge \big((n+1)^\alpha - 1\big) - \frac{1}{2^{1-\alpha}}\left\lfloor \frac{n}{2}\right\rfloor^\alpha\\
&\ge \big((n+1)^\alpha - 1\big) - \frac{1}{2^{1-\alpha}}\frac{n^\alpha}{2^\alpha}\\
&= \big((n+1)^\alpha - 1\big) - \frac{n^\alpha}{2}\\
&\ge \big((n+1)^\alpha - 1\big) - \frac{(n+1)^\alpha}{2}\\
&= \frac{(n+1)^\alpha}{2} - 1.
\end{aligned}$$

So (3.12) holds. Consequently, (3.6) holds for $r = 1$ and all $n \ge 1$.

For the inductive step, assume that (3.6) holds for a certain $r \in \mathbb{N}$ and all $n \ge 1$, we will prove that (3.6) holds for $r+1$ and all $n \ge p_{r+1} + 1$.

Observe that

$$\begin{aligned}
I_{r+1} &= \big\{k \in \mathbb{N} : \, p_i \nmid k \text{ for all } 1 \le i \le r+1\big\}\\
&= \big\{k \in \mathbb{N} : \, p_i \nmid k \text{ for all } 1 \le i \le r\big\} \backslash \big\{k \in \mathbb{N} : \, p_{r+1} | k\big\}\\
&= I_r \backslash \big(\big\{k \in \mathbb{N} : \, p_{r+1} | k\big\} \cap I_r\big),
\end{aligned}$$

and $\{k \in \mathbb{N} : p_{r+1}|k\} \cap I_r = \{jp_{r+1} : j \in I_r\}$. It follows that

$$\sum_{\substack{k=1\\k\in I_{r+1}}}^{n} \frac{1}{k^{1-\alpha}} = \sum_{\substack{k=1\\k\in I_r}}^{n} \frac{1}{k^{1-\alpha}} - \sum_{\substack{k=1\\k\in\{jp_{r+1}:j\in I_r\}}}^{n} \frac{1}{k^{1-\alpha}}$$
$$= \sum_{\substack{k=1\\k\in I_r}}^{n} \frac{1}{k^{1-\alpha}} - \frac{1}{p_{r+1}^{1-\alpha}} \sum_{\substack{j=1\\j\in I_r}}^{\lfloor n/p_{r+1}\rfloor} \frac{1}{j^{1-\alpha}}.$$

By the inductive hypothesis, (3.6) holds for r and all $n \geq 1$. Thus

$$\begin{aligned}
\alpha \sum_{\substack{k=1\\k\in I_{r+1}}}^{n} \frac{1}{k^{1-\alpha}} &\leq A_r n^\alpha + B_r - \frac{1}{p_{r+1}^{1-\alpha}}\Big(A_r\big(\Big\lfloor\frac{n}{p_{r+1}}\Big\rfloor + 1\big)^\alpha - B_r\Big) \\
&\leq A_r n^\alpha + B_r - \frac{1}{p_{r+1}^{1-\alpha}}\Big(A_r \frac{n^\alpha}{p_{r+1}^\alpha} - B_r\Big) \\
&= A_r n^\alpha\Big(1 - \frac{1}{p_{r+1}}\Big) + B_r\Big(1 + \frac{1}{p_{r+1}^{1-\alpha}}\Big) \\
&\leq A_r n^\alpha\Big(1 - \frac{1}{p_{r+1}}\Big) + 2B_r \\
&= A_{r+1} n^\alpha + B_{r+1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha \sum_{\substack{k=1\\k\in I_{r+1}}}^{n} \frac{1}{k^{1-\alpha}} &\geq A_r(n+1)^\alpha - B_r - \frac{1}{p_{r+1}^{1-\alpha}}\Big(A_r\Big\lfloor\frac{n}{p_{r+1}}\Big\rfloor^\alpha + B_r\Big) \\
&\geq A_r(n+1)^\alpha - B_r - \frac{1}{p_{r+1}^{1-\alpha}}\Big(\frac{A_r n^\alpha}{p_{r+1}^\alpha} + B_r\Big) \\
&\geq A_r(n+1)^\alpha - B_r - \frac{1}{p_{r+1}^{1-\alpha}}\Big(\frac{A_r (n+1)^\alpha}{p_{r+1}^\alpha} + B_r\Big) \\
&= A_r(n+1)^\alpha\Big(1 - \frac{1}{p_{r+1}}\Big) - B_r\Big(1 + \frac{1}{p_{r+1}^{1-\alpha}}\Big) \\
&\geq A_r(n+1)^\alpha\Big(1 - \frac{1}{p_{r+1}}\Big) - 2B_r. \\
&= A_{r+1}(n+1)^\alpha - B_{r+1}.
\end{aligned}$$

Therefore is (3.6) established. □

Why is it not possible to deduce that $\mathcal{M}(\mathcal{H}(ces_p)) = \mathcal{A}^{1/q}$ from the conclusion of Theorem 3.15, despite the fact that the result holds for all $r \in \mathbb{N}$ and the bound obtained

$$\sum_{n\in\mathbb{N}_r} |a_n| n^{-1/q} \leq \|f\|_{\mathcal{M}(\mathcal{H}(ces_p))}$$

is independent of r?

The reason, or at least one of the reasons, lays on the properties of the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ with respect to the coefficient-wise order. Consider a multiplier $f \in \mathcal{M}(\mathcal{H}(ces_p))$ with $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. For $r \in \mathbb{N}$, let $Q_r(f)$ be the Dirichlet series whose coefficients are given by

$$Q_r(f)_n := \begin{cases} a_n & n \in \mathbb{N}_r \\ 0 & n \notin \mathbb{N}_r. \end{cases}$$

The Dirichlet series $Q_r(f)$ satisfies the condition placed on the coefficients in Theorem 3.15, but, since we do not know that the multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ is solid for the coefficient-wise order, we cannot guarantee that $Q_r(f)$ is a multiplier on $\mathcal{H}(ces_p)$. Even if we could prove, for all $r \in \mathbb{N}$, that $Q_r(f) \in \mathcal{M}(\mathcal{H}(ces_p))$ and so, $Q_r(f) \in \mathcal{A}^{1/q}$ with

$$\|Q_r(f)\|_{\mathcal{A}^{1/q}} = \|Q_r(f)\|_{\mathcal{M}(\mathcal{H}(ces_p))},$$

we could neither guarantee that $\sup_r \|Q_r(f)\|_{\mathcal{M}(\mathcal{H}(ces_p))} < \infty$. This is why from Theorem 3.15 we cannot conclude that conjecture (3.5) is true. Note, however, that if $\mathcal{M}(\mathcal{H}(ces_p))$ were solid, then by Remark 3.14 we would immediately have the conjecture (3.5).

It turns out that it is more successful to carefully analyze the proof of Theorem 3.15, with the aim of removing the restriction that $a_n = 0$ for $n \notin \mathbb{N}_r$ for some $r \in \mathbb{N}$. The key point of the proof of Theorem 3.15 is that the definition of the sequence b^α (3.7), allows the critical fact, (3.9), that

$$k \in \mathbb{N} \text{ implies } k = \omega\,\gamma, \quad \omega \in \mathbb{N}_r, \gamma \in I_r,$$

and

$$(a \cdot b^\alpha)_k = a_\omega b^\alpha_\gamma.$$

A different definition of the sequence b, (3.14), allows a related critical fact, (3.16),

$$(a \cdot b)_k = a_\omega h(r),$$

which leads to the identification of the multiplier space.

Theorem 3.17. *For $1 < p < \infty$ and $1/p + 1/q = 1$, we have*

$$\mathcal{M}(\mathcal{H}(ces_p)) = \mathcal{A}^{1/q}$$

with equality of norms.

Proof. Let $f \in \mathcal{M}(\mathcal{H}(ces_p))$ with $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Recall that p_r denotes the r-th prime number. Since by the Prime Number Theorem

$$\lim_{r\to\infty} \frac{p_r}{r\log r} = 1,$$

for each $m \in \mathbb{N}$ there exists $r_m \in \mathbb{N}$ with $r_m > m$ such that

$$1 - \frac{1}{m} \leq \frac{p_r}{r\log r} \leq 1 + \frac{1}{m}, \quad \text{for all } r \geq r_m,$$

and so

$$\Big(1 - \frac{1}{m}\Big) r\log r \leq p_r \leq \Big(1 + \frac{1}{m}\Big) r\log r, \quad \text{for all } r \geq r_m. \tag{3.13}$$

For a fixed $m \in \mathbb{N}$, consider the sequence $b = (b_n)_{n=1}^{\infty}$ defined by

$$b_n := \begin{cases} h(r) & n = p_r \text{ with } r \geq r_m, \\ 0 & \text{in other case,} \end{cases} \tag{3.14}$$

for an appropriate positive function h to be determined later. Let $g(s) := \sum_{n=1}^{\infty} b_n n^{-s}$. Then,

$$\begin{aligned} \|g\|^p_{\mathcal{H}(ces_p)} = \|b\|^p_{ces_p} &= \sum_{n=1}^{\infty} \frac{1}{n^p} \Big(\sum_{k=1}^{n} |b_k| \Big)^p \\ &= \sum_{n=p_{r_m}}^{\infty} \frac{1}{n^p} \Big(\sum_{\substack{r=r_m \\ p_r \leq n}}^{\infty} h(r) \Big)^p. \end{aligned} \tag{3.15}$$

Note that for each $k = \omega\, p_r$ with $\omega \in \mathcal{P}_m$ and $r \geq r_m$, it follows that

$$(a \cdot b)_k = \sum_{j|k} a_j b_{\frac{k}{j}} = a_\omega b_{p_r} = a_\omega h(r). \tag{3.16}$$

Indeed, if $j|k$ and $j \neq \omega$ we have that $k/j \neq p_r$ for all $r \geq r_m$ and so $b_{k/j} = 0$.

For the value $m \in \mathbb{N}$ fixed above, consider the subset of $\mathbb{N}$ given by

$$\mathcal{P}_m := \Big\{ \prod_{i=1}^{r_m - 1} p_i^{\alpha_i} \,:\, 0 \leq \alpha_i \leq m \ \text{ for all } 1 \leq i \leq r_m - 1 \Big\}.$$

Since $\omega p_r = \hat{\omega} p_{\hat{r}}$ with $\omega, \hat{\omega} \in \mathcal{P}_m$ and $r, \hat{r} \geq r_m$ implies that $\omega = \hat{\omega}$ and $p_r = p_{\hat{r}}$, it follows that the set

$$\mathcal{O}_m = \bigcup_{\omega \in \mathcal{P}_m} \omega \cdot \{p_r \,:\, r \geq r_m\}$$

is a finite union of disjoint sets. Then, for any $n_m \geq p_{r_m}^{mr_m+1}$, we have that

$$\begin{aligned}
\|fg\|^p_{\mathcal{H}(ces_p)} = \|a\cdot b\|^p_{ces_p} &= \sum_{n=1}^{\infty} \frac{1}{n^p}\Big(\sum_{k=1}^{n} |(a\cdot b)_k|\Big)^p \\
&\geq \sum_{n=n_m}^{\infty} \frac{1}{n^p}\Big(\sum_{\substack{k=1\\ k\in\mathcal{O}_m}}^{n} |(a\cdot b)_k|\Big)^p \\
&= \sum_{n=n_m}^{\infty} \frac{1}{n^p}\Big(\sum_{\omega\in\mathcal{P}_m} \sum_{\substack{r=r_m\\ p_r\leq \frac{n}{\omega}}}^{\infty} |(a\cdot b)_{\omega p_r}|\Big)^p \\
&= \sum_{n=n_m}^{\infty} \frac{1}{n^p}\Big(\sum_{\omega\in\mathcal{P}_m} |a_\omega| \sum_{\substack{r=r_m\\ p_r\leq \frac{n}{\omega}}}^{\infty} h(r)\Big)^p. \qquad (3.17)
\end{aligned}$$

Note that $\omega \leq p_{r_m}^{mr_m}$ whenever $\omega\in\mathcal{P}_m$ and so $n/\omega \geq p_{r_m}$ for $n\geq n_m$.

We use the next lemma which will be proved afterwards. Set $1/2q < \alpha < 1/q$ and

$$\phi(x) := (x\log x)^\alpha, \quad x\in[1,\infty).$$

Note that the derivative ϕ' of ϕ is positive. Define $h := \phi'$.

Lemma 3.18. *There exists x_q large enough such that*

$$\Big(\frac{\beta m}{m+1}\Big)^\alpha - \big(r_m\log r_m\big)^\alpha \leq \sum_{\substack{r=r_m\\ p_r\leq\beta}} h(r) \leq \Big(\frac{\beta m}{m-1}\Big)^\alpha \qquad (3.18)$$

whenever $r_m \geq x_q$ and $\beta \geq p_{r_m}$. Note that $\big(\frac{\beta m}{m+1}\big)^\alpha - \big(r_m\log r_m\big)^\alpha \geq 0$ precisely when $r_m\log r_m(1+\frac{1}{m}) \leq \beta$.

Assume that $r_m \geq x_q$. From (3.15) and Lemma 3.18 we have that

$$\begin{aligned}
\|g\|^p_{\mathcal{H}(ces_p)} = \|b\|^p_{ces_p} &\leq \sum_{n=p_{r_m}}^{\infty} \frac{1}{n^p}\Big(\frac{nm}{m-1}\Big)^{p\alpha} \\
&= \Big(\frac{m}{m-1}\Big)^{p\alpha} \sum_{n=p_{r_m}}^{\infty} \frac{1}{n^{p(1-\alpha)}} \qquad (3.19)\\
&\leq \Big(\frac{m}{m-1}\Big)^{p\alpha} \frac{1}{\big(p(1-\alpha)-1\big)(p_{r_m}-1)^{(p(1-\alpha)-1)}}.
\end{aligned}$$

On the other hand, for $n_m = 2r_m^2 p_{r_m}^{mr_m+2q}$, from (3.17) and Lemma 3.18 it follows that

$$\begin{aligned}\|fg\|^p_{\mathcal{H}(ces_p)} &= \|a\cdot b\|^p_{ces_p}\\ &\geq \sum_{n=n_m}^{\infty}\frac{1}{n^p}\left(\sum_{\omega\in\mathcal{P}_m}|a_\omega|\left(\left(\frac{nm}{\omega(m+1)}\right)^\alpha-\left(r_m\log r_m\right)^\alpha\right)\right)^p.\end{aligned}$$

Note, for $n\geq n_m$ and $\omega\in\mathcal{P}_m$, that

$$\begin{aligned}\left(\frac{nm}{\omega(m+1)}\right)^\alpha-\left(r_m\log r_m\right)^\alpha &= \left(\frac{nm}{\omega(m+1)}\right)^\alpha\left(1-\left(\frac{\omega(m+1)r_m\log r_m}{nm}\right)^\alpha\right)\\ &\geq \left(\frac{nm}{\omega(m+1)}\right)^\alpha\left(1-\frac{1}{p_{r_m}}\right),\end{aligned}$$

since

$$\begin{aligned}\frac{\omega(m+1)r_m\log r_m}{m} &\leq \frac{p_{r_m}^{mr_m}(m+1)r_m\log r_m}{m}\\ &\leq \frac{2r_m^2 p_{r_m}^{mr_m+2q}}{p_{r_m}^{2q}}\\ &= \frac{n_m}{p_{r_m}^{2q}}\\ &\leq \frac{n_m}{p_{r_m}^{1/\alpha}}\\ &\leq \frac{n}{p_{r_m}^{1/\alpha}}.\end{aligned}$$

Then,

$$\begin{aligned}\|fg\|^p_{\mathcal{H}(ces_p)} &= \|a\cdot b\|^p_{ces_p}\\ &\geq \sum_{n=n_m}^{\infty}\frac{1}{n^p}\left(\sum_{\omega\in\mathcal{P}_m}|a_\omega|\left(\frac{nm}{\omega(m+1)}\right)^\alpha\left(1-\frac{1}{p_{r_m}}\right)\right)^p\\ &= \left(1-\frac{1}{p_{r_m}}\right)^p\left(\frac{m}{m+1}\right)^{p\alpha}\sum_{n\geq n_m}\frac{1}{n^{p(1-\alpha)}}\left(\sum_{\omega\in\mathcal{P}_m}\frac{|a_\omega|}{\omega^\alpha}\right)^p \qquad (3.20)\\ &\geq \left(1-\frac{1}{p_{r_m}}\right)^p\left(\frac{m}{m+1}\right)^{p\alpha}\frac{1}{\left(p(1-\alpha)-1\right)n_m^{(p(1-\alpha)-1)}}\left(\sum_{\omega\in\mathcal{P}_m}\frac{|a_\omega|}{\omega^\alpha}\right)^p.\end{aligned}$$

From (3.19) and (3.20) it follows that

$$\begin{aligned}
\|f\|^p_{\mathcal{M}(\mathcal{H}(ces_p))} &\geq \frac{\|a\cdot b\|^p_{ces_p}}{\|b\|^2_{ces_2}} \\
&\geq \frac{(1-\frac{1}{p_{r_m}})^p(\frac{m}{m+1})^{p\alpha}\frac{1}{(p(1-\alpha)-1)n_m^{(p(1-\alpha)-1)}}\left(\sum_{\omega\in\mathcal{P}_m}\frac{|a_\omega|}{\omega^\alpha}\right)^p}{(\frac{m}{m-1})^{p\alpha}\frac{1}{(p(1-\alpha)-1)(p_{r_m}-1)^{(p(1-\alpha)-1)}}} \\
&= \frac{(p_{r_m}-1)^{p(2-\alpha)-1}}{p^p_{r_m}}\left(\frac{m-1}{m+1}\right)^{p\alpha}\frac{1}{n_m^{(p(1-\alpha)-1)}}\left(\sum_{\omega\in\mathcal{P}_m}\frac{|a_\omega|}{\omega^\alpha}\right)^p
\end{aligned}$$

Taking limit as $\alpha\to(1/q)^-$ we have

$$\|f\|^p_{\mathcal{M}(\mathcal{H}(ces_p))}\geq\frac{(p_{r_m}-1)^p}{p^p_{r_m}}\left(\frac{m-1}{m+1}\right)^{p/q}\left(\sum_{\omega\in\mathcal{P}_m}\frac{|a_\omega|}{\omega^{1/q}}\right)^p.$$

Finally, making $m\to\infty$ we conclude

$$\|f\|^p_{\mathcal{M}(\mathcal{H}(ces_p))}\geq\left(\sum_{\omega\in\mathbb{N}}\frac{|a_\omega|}{\omega^{1/q}}\right)^p.$$

□

Proof of Lemma 3.18. The left-hand side of (3.18) is

$$\left(\frac{\beta m}{m+1}\right)^\alpha-\left(r_m\log r_m\right)^\alpha\leq\sum_{\substack{r=r_m\\p_r\leq\beta}}h(r),$$

for $h:=\phi'$ where $\phi(x):=(x\log x)^\alpha$ on $[1,\infty)$. Since h is positive, it follows that the inequality holds trivially when its left-hand side is negative, that is, when $r_m\log r_m(1+\frac{1}{m})>\beta$. Thus, we only have to consider the case when $r_m\log r_m(1+\frac{1}{m})\leq\beta$, that is, when $\gamma_m\leq\beta$, where we denote $\gamma_m:=r_m\log r_m(1+\frac{1}{m})$. Note, by (3.13), that $p_{r_m}\leq\gamma_m$.

Let W be the Lambert function on $(0,\infty)$ defined by $W(x)e^{W(x)}=x$. Then

$$\frac{x}{W(x)}\log\left(\frac{x}{W(x)}\right)=x. \tag{3.21}$$

We first show that

$$\sum_{r=r_m}^{\left\lfloor\frac{\frac{\beta m}{m+1}}{W(\frac{\beta m}{m+1})}\right\rfloor}h(r)\leq\sum_{\substack{r=r_m\\p_r\leq\beta}}h(r)\leq\sum_{r=r_m}^{\left\lfloor\frac{\frac{\beta m}{m-1}}{W(\frac{\beta m}{m-1})}\right\rfloor}h(r) \tag{3.22}$$

whenever $\beta \geq p_{r_m}$, for the right-hand inequality, and $\beta \geq \gamma_m$, for the left-hand inequality.

Consider the right-hand inequality of (3.22). For each $r \geq r_m$ such that $p_r \leq \beta$, by (3.13) and (3.21), we have that

$$r \log r \leq \frac{\beta m}{m-1} = \frac{\frac{\beta m}{m-1}}{W\left(\frac{\beta m}{m-1}\right)} \log \left(\frac{\frac{\beta m}{m-1}}{W\left(\frac{\beta m}{m-1}\right)} \right).$$

Since $x \log x$ is an increasing injective function on $[1, \infty)$, it follows that

$$r \leq \frac{\frac{\beta m}{m-1}}{W\left(\frac{\beta m}{m-1}\right)}.$$

Hence, the right-hand inequality of (3.22) holds.

Consider now the left-hand inequality of (3.22). For each r satisfying

$$r \leq \frac{\frac{\beta m}{m+1}}{W\left(\frac{\beta m}{m+1}\right)},$$

from (3.13) and (3.21) we have that

$$\begin{aligned} p_r &\leq r \log r \left(1 + \frac{1}{m}\right) \\ &\leq \frac{\frac{\beta m}{m-1}}{W\left(\frac{\beta m}{m-1}\right)} \log \left(\frac{\frac{\beta m}{m-1}}{W\left(\frac{\beta m}{m-1}\right)} \right) \left(1 + \frac{1}{m}\right) \\ &= \frac{\beta m}{m+1} \left(1 + \frac{1}{m}\right) \\ &= \beta \end{aligned}$$

and so the left-hand inequality of (3.22) holds. Note that $r_m \leq \frac{\beta m}{m+1} / W(\frac{\beta m}{m+1})$ if and only if $\gamma_m \leq \beta$.

Let x_q be sufficiently large so that h is decreasing on $[x_q, \infty)$. Such value x_q exists. Indeed,

$$h(x) = \phi'(x) = \alpha (x \log x)^{\alpha - 1} (\log x + 1)$$

and so

$$\begin{aligned} h'(x) &= \alpha(\alpha-1)(x\log x)^{\alpha-2}(\log x+1)^2 + \alpha(x\log x)^{\alpha-1}\frac{1}{x} \\ &= \alpha(x\log x)^{\alpha-2}\Big((\alpha-1)(\log x+1)^2 + \frac{x\log x}{x}\Big) \\ &= \alpha(x\log x)^{\alpha-2}(\log x+1)^2\Big(\alpha - 1 + \frac{\log x}{(\log x+1)^2}\Big) \\ &\leq \alpha(x\log x)^{\alpha-2}(\log x+1)^2\Big(\frac{1}{q} - 1 + \frac{\log x}{(\log x+1)^2}\Big). \end{aligned}$$

Since $\lim_{x\to\infty}\log x(\log x+1)^{-2} = 0$ and $1/q-1<0$, there exists x_q such that $h'(x) \leq 0$ for all $x \geq x_q - 1$.

Then, for every $M \geq N \geq x_q$ it follows that

$$\begin{aligned} \sum_{r=N}^{M} h(r) &\leq \sum_{r=N}^{M}\int_{r-1}^{r} h(x)\,dx \\ &= \int_{N-1}^{M} h(x)\,dx \\ &= \phi(M) - \phi(N-1) \\ &\leq \phi(M) \end{aligned}$$

and

$$\begin{aligned} \sum_{r=N}^{M} h(r) &\geq \sum_{r=N}^{M}\int_{r}^{r+1} h(x)\,dx \\ &= \int_{N}^{M+1} h(x)\,dx \\ &= \phi(M+1) - \phi(N). \end{aligned}$$

Hence, from (3.22), we have

$$\phi\left(\left\lfloor\frac{\frac{\beta m}{m+1}}{W(\frac{\beta m}{m+1})}\right\rfloor + 1\right) - \phi(r_m) \leq \sum_{\substack{r=r_m \\ p_r\leq\beta}}^{\infty} h(r) \leq \phi\left(\left\lfloor\frac{\frac{\beta m}{m-1}}{W(\frac{\beta m}{m-1})}\right\rfloor\right) \tag{3.23}$$

whenever $r_m \geq x_q$ and whenever $\beta \geq p_{r_m}$, for the right-hand inequality, and $\beta \geq \gamma_m$, for the left-hand inequality. From (3.21) we have

$$\begin{aligned}\phi\left(\left\lfloor\frac{\frac{\beta m}{m-1}}{W(\frac{\beta m}{m-1})}\right\rfloor\right) &\leq \phi\left(\frac{\frac{\beta m}{m-1}}{W(\frac{\beta m}{m-1})}\right)\\ &= \left(\frac{\frac{\beta m}{m-1}}{W(\frac{\beta m}{m-1})}\log\left(\frac{\frac{\beta m}{m-1}}{W(\frac{\beta m}{m-1})}\right)\right)^{\alpha}\\ &= \left(\frac{\beta m}{m-1}\right)^{\alpha},\end{aligned}$$

and, similarly,

$$\phi\left(\left\lfloor\frac{\frac{\beta m}{m+1}}{W(\frac{\beta m}{m+1})}\right\rfloor + 1\right) \geq \phi\left(\frac{\frac{\beta m}{m+1}}{W(\frac{\beta m}{m+1})}\right) = \left(\frac{\beta m}{m+1}\right)^{\alpha},$$

it follows that

$$\left(\frac{\beta m}{m+1}\right)^{\alpha} - \left(r_m \log r_m\right)^{\alpha} \leq \sum_{\substack{r=r_m\\ p_r\leq\beta}} h(r) \leq \left(\frac{\beta m}{m-1}\right)^{\alpha},$$

that is, (3.18) holds. □

As a consequence of Theorem 3.17, we can describe the solid core of the multiplier algebra of $\mathcal{H}([\mathcal{C},\ell^p])$.

Proposition 3.19. *The space $\mathcal{A}^{1/q}$ is the solid core of $\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$.*

Proof. From Proposition 3.9 we have that $\mathcal{A}^{1/q} \subset \mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$. Let $\mathcal{E}$ be a solid subspace of $\mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$. For every $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{E}$, taking $h(s) := \sum_{n=1}^{\infty} |a_n| n^{-s} \in \mathcal{E} \subset \mathcal{M}(\mathcal{H}([\mathcal{C},\ell^p]))$, by Proposition 3.10, it follows that $h \in \mathcal{M}(\mathcal{H}(ces_p)) = \mathcal{A}^{1/q}$ and so $f \in \mathcal{A}^{1/q}$. Then, $\mathcal{E} \subset \mathcal{A}^{1/q}$. □

3.4 Further facts on multipliers on $\mathcal{H}(ces_p)$

We study the compactness of the multipliers on $\mathcal{H}(ces_p)$. It turns out that there is no other compact multiplier than zero.

Theorem 3.20. *Let $f \in \mathcal{M}(\mathcal{H}(ces_p))$. Suppose that the operator*

$$g \in \mathcal{H}(ces_p) \mapsto M_f(g) := fg \in \mathcal{H}(ces_p)$$

is compact. Then $f = 0$.

Proof. Consider the sequence $\{m^{1/q}m^{-s}\}_{m=1}^{\infty}$ in $\mathcal{H}(ces_p)$. It is bounded as, for $m \geq 2$, we have that

$$\begin{aligned}
\|m^{1/q}m^{-s}\|_{\mathcal{H}(ces_p)} &= m^{1/q}\|e^m\|_{ces_p} \\
&= m^{1/q}\Big(\sum_{n=m}^{\infty}\frac{1}{n^p}\Big)^{1/p} \\
&\leq m^{1/q}\Big(\frac{1}{(p-1)(m-1)^{p-1}}\Big)^{1/p} \\
&= \frac{m^{1/q}}{(p-1)^{1/p}(m-1)^{1/q}} \\
&\leq \frac{2^{1/q}}{(p-1)^{1/p}}.
\end{aligned}$$

Then, there exists a subsequence $\{m_k^{1/q}m_k^{-s}\}_{k=1}^{\infty}$ such that $\{M_f(m_k^{1/q}m_k^{-s})\}_{k=1}^{\infty}$ converges to some $g \in \mathcal{H}(ces_p)$. For $s_0 \in \mathbb{C}_{1/q}$, since the point evaluation δ_{s_0} is bounded on $\mathcal{H}(ces_p)$, we have

$$\delta_{s_0}\Big(M_f(m_k^{1/q}m_k^{-s})\Big) \xrightarrow[k\to\infty]{} \delta_{s_0}(g) = g(s_0).$$

On the other hand,

$$\delta_{s_0}\Big(M_f(m_k^{1/q}m_k^{-s})\Big) = f(s_0)m_k^{1/q-s_0} \xrightarrow[k\to\infty]{} 0.$$

Thus, $g = 0$. Hence, $\{M_f(m_k^{1/q}m_k^{-s})\}_{k=1}^{\infty}$ converges to zero in the norm of $\mathcal{H}(ces_p)$.

We estimate from below $\|M_f(m_k^{1/q}m_k^{-s})\|_{\mathcal{H}(ces_p)} = m_k^{1/q}\|M_f(m_k^{-s})\|_{\mathcal{H}(ces_p)}$. Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. We have seen in the proof of Proposition 3.4 that

$$\begin{aligned}
\|M_f(m^{-s})\|_{\mathcal{H}(ces_p)}^p &= \|m^{-s}f\|_{\mathcal{H}(ces_p)}^p \\
&= \sum_{j=1}^{\infty}\Big(\sum_{i=1}^{j}|a_i|\Big)^p\sum_{n=jm}^{(j+1)m-1}\frac{1}{n^p}.
\end{aligned}$$

Since

$$\sum_{n=jm}^{(j+1)m-1}\frac{1}{n^p} \geq \frac{m}{\big((j+1)m-1\big)^p} \geq \frac{m}{(j2m)^p},$$

it follows that

$$\|M_f(m^{-s})\|_{\mathcal{H}(ces_p)}^p \geq \frac{m}{(2m)^p}\sum_{j=1}^{\infty}\frac{1}{j^p}\Big(\sum_{i=1}^{j}|a_i|\Big)^p = \frac{m}{(2m)^p}\|f\|_{\mathcal{H}(ces_p)}^p.$$

Then,

$$\|M_f(m_k^{1/q}m_k^{-s})\|_{\mathcal{H}(ces_p)} \geq m_k^{1/q}\frac{m_k^{1/p}}{2m_k}\|f\|_{\mathcal{H}(ces_p)} = \frac{1}{2}\|f\|_{\mathcal{H}(ces_p)}.$$

Taking $k \to \infty$ we have that $\|f\|_{\mathcal{H}(ces_p)} \leq 0$ and so, $f = 0$. $\square$

We end this chapter discussing how "close" is the space $\mathcal{H}(ces_p)$ to its multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$, which we know from Theorem 3.17 that it is the space $\mathcal{A}^{1/q}$. Let us first note that $\mathcal{A}^{1/q} \subsetneq \mathcal{H}(ces_p)$. Indeed, in other case the point evaluation at $s_0 = 1/q$, $\delta_{1/q}$, which belongs to the dual space of $\mathcal{A}^{1/q}$, belong to the dual space of $\mathcal{H}(ces_p)$. This would imply that $(n^{-1/q})_{n=1}^\infty \in d(q)$ (see (2.2) in Section 2.1), that is, $(n^{-1/q})_{n=1}^\infty \in \ell^q$, which is a contradiction.

The multiplier algebra $\mathcal{M}(\mathcal{H}(ces_p))$ is "close" to $\mathcal{H}(ces_p)$ in the sense shown by the following example. For $f \in \mathcal{H}(ces_p)$ with $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ and $\varepsilon > 0$, set

$$f_{a(\varepsilon)}(s) := \sum_{n=1}^\infty a_n n^{-\varepsilon} n^{-s}.$$

Proposition 2.12 shows that $\sigma_a(\mathcal{H}(ces_p)) = 1/q$. Then

$$\sum_{n=1}^\infty \frac{|a_n n^{-\varepsilon}|}{n^{1/q}} = \sum_{n=1}^\infty \frac{|a_n|}{n^{1/q+\epsilon}} < \infty,$$

Consequently, $f_{a(\varepsilon)} \in \mathcal{A}^{1/q} = \mathcal{M}(\mathcal{H}(ces_p))$, that is, $f_{a(\varepsilon)}$ is a multiplier on $\mathcal{H}(ces_p)$.

The question arises: For which sequences $(b_n)_{n=1}^\infty$ it is the case that $\sum_{n=1}^\infty a_n b_n n^{-s}$ is a multiplier on $\mathcal{H}(ces_p)$ whenever $\sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}(ces_p)$?

Theorem 3.21. *Let $\varphi : [1, \infty) \to \mathbb{R}$ be a bounded differentiable function satisfying*

$$\sum_{n=1}^\infty \max_{n \leq x \leq n+1} \left| \varphi'(x) - \frac{\varphi(x)}{qx} \right| < \infty. \tag{3.24}$$

For every $f \in \mathcal{H}(ces_p)$ with $f(s) = \sum_{n=1}^\infty a_n n^{-s}$, consider the Dirichlet series

$$f_{a(\varphi)}(s) := \sum_{n=1}^\infty a_n \varphi(n) n^{-s}.$$

Then $f_{a(\varphi)} \in \mathcal{A}^{1/q}$, and so $f_{a(\varphi)}$ is a multiplier on $\mathcal{H}(ces_p)$.

Proof. For $N \geq 1$, set $A_N := |a_1| + \ldots + |a_N|$ and $A_0 := 0$. Abel's summation formula gives

$$\begin{aligned}\sum_{n=1}^{N} \left| \frac{a_n \varphi(n)}{n^{1/q}} \right| &= \sum_{n=1}^{N} (A_n - A_{n-1}) \frac{|\varphi(n)|}{n^{1/q}} \\ &= \frac{A_N |\varphi(N)|}{N^{1/q}} + \sum_{n=1}^{N-1} A_n \left(\frac{|\varphi(n)|}{n^{1/q}} - \frac{|\varphi(n+1)|}{(n+1)^{1/q}} \right).\end{aligned}$$

Since φ is differentiable,

$$\begin{aligned}\left| \frac{|\varphi(n)|}{n^{1/q}} - \frac{|\varphi(n+1)|}{(n+1)^{1/q}} \right| &\leq \left| \frac{\varphi(n)}{n^{1/q}} - \frac{\varphi(n+1)}{(n+1)^{1/q}} \right| \\ &\leq \max_{n \leq x \leq n+1} \left| \frac{d}{dx} \left(\varphi(x) x^{-1/q} \right) \right| \\ &= \max_{n \leq x \leq n+1} \left| \varphi'(x) x^{-1/q} - \frac{1}{q} \varphi(x) x^{-1/q-1} \right| \\ &\leq \max_{n \leq x \leq n+1} n^{-1/q} \left| \varphi'(x) - \frac{\varphi(x)}{qx} \right|\end{aligned}$$

Thus,

$$\sum_{n=1}^{N} \left| \frac{a_n \varphi(n)}{n^{1/q}} \right| \leq \frac{A_N |\varphi(N)|}{N^{1/q}} + \sum_{n=1}^{N} \frac{A_n}{n^{1/q}} \cdot \max_{n \leq x \leq n+1} \left| \varphi'(x) - \frac{\varphi(x)}{qx} \right|.$$

Recall that, according to Proposition 2.2, the growth of the sequence $a \in ces_p$ is restrained by

$$\lim_{n \to \infty} \frac{1}{n^{1/q}} \sum_{k=1}^{n} |a_k| = 0.$$

It follows that the sequence $(A_n n^{-1/q})_{n=1}^{\infty}$ is bounded by a constant $K > 0$, and $A_N \varphi(N) N^{-1/q} \to 0$ as $N \to \infty$, since φ is bounded. Then

$$\sum_{n=1}^{\infty} \frac{|a_n \varphi(n)|}{n^{1/q}} \leq K \sum_{n=1}^{\infty} \max_{n \leq x \leq n+1} \left| \varphi'(x) - \frac{\varphi(x)}{qx} \right| < \infty.$$

□

Example 3.22. For $\varphi(x) := \log^{-\alpha} x$, for $\alpha > 1$ with $\alpha > 1$ considered on $[2, \infty)$, we have

$$\max_{n \leq x \leq n+1} \left| \varphi'(x) - \frac{\varphi(x)}{qx} \right| \leq \frac{\alpha + 1/q}{n \log^{\alpha} n}, \quad n \geq 2.$$

Thus, we are in the conditions of Proposition 3.21, and so for every $\sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}(ces_p)$ we have that

$$\sum_{n=2}^{\infty} \frac{a_n}{\log^{\alpha} n} n^{-s}$$

is a multiplier on $\mathcal{H}(ces_p)$.

Remark 3.23. Proposition 3.21 can alternatively be viewed as follows: under condition (3.24) the sequence $(\varphi(n))_{n=1}^{\infty}$ is a *multiplier sequence* from $\mathcal{H}(ces_p)$ into $\mathcal{A}^{1/q}$, in the sense that

$$\sum_{n=1}^{\infty} a_n \varphi(n) n^{-s} \in \mathcal{A}^{1/q} \quad \text{for each} \quad \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}(ces_p).$$

Chapter 4

The Cesàro operator acting on Dirichlet series

We end this memoire with a brief comment concerning the Cesàro averaging operator $\mathcal{C}$ on sequences,

$$a = (a_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}} \longmapsto \mathcal{C}(a) := \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}.$$

The aim is to study the Cesàro operator when acting on different spaces of Dirichlet series via their sequence of coefficients,

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \longmapsto \mathcal{C}(f)(s) := \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) n^{-s}, \quad s \in \mathbb{C}. \tag{4.1}$$

For example, Hardy's inequality for $p = 2$,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} |a_k|\right)^2 < 4\sum_{n=1}^{\infty} |a_n|^2,$$

can be interpreted as asserting that $\mathcal{C}$ maps $\mathcal{H}$ into $\mathcal{H}$ boundedly,

$$\|\mathcal{C}(f)\|_{\mathcal{H}} \leq 2\|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

In fact, the construction of the sequence space ces_2 guarantees that $\mathcal{C}$ maps $\mathcal{H}(ces_2)$ into $\mathcal{H}$ boundedly,

$$\|\mathcal{C}(f)\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}(ces_2)}, \quad f \in \mathcal{H}(ces_2).$$

What happens, for example, for the spaces $\mathcal{H}^p$, $1 \leq p < \infty$, defined by Bayart, in the case when $p \neq 2$? For studying this last question (and other similar ones) we present an integral formula for the action of the Cesàro operator on Dirichlet series analogous to the classical integral formula available for the action of the Cesàro operator on Taylor series, see (4.2) below. The deduction of such formula is shown in (4.5) and (4.6) below. Unfortunately, the study was not continued further.

Formula for the Cesàro operator acting on Taylor series

The Cesàro operator acting on the space $H(\mathbb{D})$ of analytic functions on the open unit disk $\mathbb{D}$ is $\mathcal{C}\colon H(\mathbb{D}) \to H(\mathbb{D})$, defined by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \longmapsto \mathcal{C}(f)(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

Indeed, if the Taylor series for f defines an analytic function on $\mathbb{D}$, then the Taylor series for $\mathcal{C}(f)$ also defines an analytic function on $\mathbb{D}$.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ and $z_0 \in \mathbb{D}$, we have that

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z_0^n &= \sum_{k=0}^{\infty} a_k \sum_{n=k}^{\infty} \frac{z_0^n}{n+1} \\
&= \frac{1}{z_0} \sum_{k=0}^{\infty} a_k \sum_{n=k}^{\infty} \frac{z_0^{n+1}}{n+1} \\
&= \frac{1}{z_0} \sum_{k=0}^{\infty} a_k \sum_{n=k}^{\infty} \int_0^{z_0} \xi^n \, d\xi \\
&= \frac{1}{z_0} \int_0^{z_0} \sum_{k=0}^{\infty} a_k \sum_{n=k}^{\infty} \xi^n \, d\xi \\
&= \frac{1}{z_0} \int_0^{z_0} \frac{1}{1-\xi} \sum_{k=0}^{\infty} a_k \xi^k \, d\xi.
\end{aligned}$$

Then, we arrive to the integral expression

$$\mathcal{C}(f)(z_0) = \frac{1}{z_0} \int_0^{z_0} \frac{f(\xi)}{1-\xi} \, d\xi, \quad z_0 \in \mathbb{D}, \tag{4.2}$$

which has many applications in the study of the Cesàro operator.

Formula for the Cesàro operator on Dirichlet series

We look for an integral formula similar to (4.2) for the Cesàro operator acting on Dirichlet series in the sense of (4.1). First note, for $\sigma > \max\{\sigma_a(f), 0\}$, that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|n^{-\sigma} &\leq \sum_{n=1}^{\infty}\frac{1}{n^{1+\sigma}}\sum_{k=1}^{n}|a_k| \\
&= \sum_{k=1}^{\infty}|a_k|\sum_{n=k}^{\infty}\frac{1}{n^{1+\sigma}} \\
&\leq |a_1|\left(1+\frac{1}{\sigma}\right)+\frac{1}{\sigma}\sum_{k=2}^{\infty}\frac{|a_k|}{(k-1)^{\sigma}} \\
&\leq |a_1|\left(1+\frac{1}{\sigma}\right)+\frac{2^{\sigma}}{\sigma}\sum_{k=2}^{\infty}\frac{|a_k|}{k^{\sigma}} < \infty.
\end{aligned}$$

Then, $\sigma_a(\mathcal{C}(f)) \leq \max\{\sigma_a(f), 0\}$. In particular $\mathcal{C}\colon \mathcal{D} \to \mathcal{D}$, as $\sigma_a(f) \leq 1 + \sigma_c(f)$ (Theorem 1.12).

Following a suggestion by Prof. Juan Arias de Reyna, we will use the Perron-Landau formulae: for $c, x > 0$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}\,ds = \begin{cases} 0 & \text{if } x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1, \end{cases}$$

where for $x = 1$ the integral is a Cauchy principal value, see [36, Theorem G, p. 75].

For a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$, $c > \max\{\sigma_a(f), 0\}$ and $x > 0$, the Perron-Landau formula gives

$$\sum_{n<x} a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}\,ds \tag{4.3}$$

when $x \notin \mathbb{N}$ and

$$\sum_{n<x} a_n + \frac{a_x}{2} = \frac{1}{2\pi i}\ \text{p.v.}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}\,ds \tag{4.4}$$

when $x \in \mathbb{N}$.

Then, via formula (4.3), for $s_0 \in \mathbb{C}_c$ and $0 < r < 1$, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)n^{-s_0} &= \sum_{n=1}^{\infty}\frac{1}{n^{1+s_0}}\sum_{k<n+r}a_k \\
&= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n^{1+s_0}}\int_{c-i\infty}^{c+i\infty} f(s)\frac{(n+r)^s}{s}\,ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left(\sum_{n=1}^{\infty}\frac{(n+r)^s}{n^{s_0+1}}\right)\frac{ds}{s}.
\end{aligned}$$

We arrive to

$$\mathcal{C}(f)(s_0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\left(\sum_{n=1}^{\infty}\frac{(n+r)^s}{n^{s_0+1}}\right)\frac{ds}{s} \tag{4.5}$$

whenever $\Re(s_0) > c > \max\{\sigma_a(f), 0\}$ and $0 < r < 1$.

We can follow the same steps using (4.4) instead of (4.3), to obtain, for $s_0 \in \mathbb{C}_c$,

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)n^{-s_0} &= \sum_{n=1}^{\infty}\frac{1}{n^{1+s_0}}\left(\sum_{k<n} a_k + \frac{a_n}{2}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_n}{n^{1+s_0}} \\
&= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n^{1+s_0}}\ \text{p.v.}\int_{c-i\infty}^{c+i\infty} f(s)\frac{n^s}{s}\,ds + \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_n}{n^{1+s_0}} \\
&= \frac{1}{2\pi i}\ \text{p.v.}\int_{c-i\infty}^{c+i\infty} f(s)\left(\sum_{n=1}^{\infty}\frac{n^s}{n^{s_0+1}}\right)\frac{ds}{s} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{a_n}{n^{1+s_0}}.
\end{aligned}$$

We arrive to

$$\mathcal{C}(f)(s_0) = \frac{1}{2\pi i}\ \text{p.v.}\int_{c-i\infty}^{c+i\infty} f(s)\left(\sum_{n=1}^{\infty}\frac{n^s}{n^{s_0+1}}\right)\frac{ds}{s} + \frac{f(1+s_0)}{2} \tag{4.6}$$

whenever $\Re(s_0) > c > \max\{\sigma_a(f), 0\}$.

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